REPORT No. 164

THE INERTIA COEFFICIENTS OF AN AIRSHIP IN A FRICTIONLESS FLUID

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SUMMARY.

The following investigation of the apparent inertia of an airship hull was made at the request of the National Advisory Committee for Aeronautics. The exact solution of the aerodynamical problem has been studied for hulls of various shapes and special attention has been given to the case of an ellipsoidal hull. In order that the results for this last case may be readily adapted to other cases, they are expressed in terms of the area and perimeter of the largest cross section perpendicular to the direction of motion by means of a formula involving a coefficient K which varies only slowly when the shape of the hull is changed, being *0.637* for a circular or elliptic disk, *0.5* for a sphere, and about *0.25* for a spheroid of fineness ratio *7*. For rough purposes it is sufficient to employ the coefficients, originally found for ellipsoids, for hulls otherwise shaped. When more exact values of the inertia are needed, estimates may be based on a study of the way in which K varies with different characteristics and for such a study the new coefficient possesses some advantages over one which is defined with reference to the volume of fluid displaced.

The case of rotation of an airship hull has been investigated also and a coefficient has been defined with the same advantages as the corresponding coefficient for rectilinear motion.

I. INTRODUCTION.

It follows from Green's analysis that when an ellipsoidal body moves in an infinite incompressible inviscid fluid in such a way that the flow is everywhere of the irrotational, continuous Eulerian type, the kinetic energy of the fluid produces an apparent increase in the mass and moments of inertia of the body. The terms mass and moment of inertia are used here in a generalized sense because it appears that the apparent mass is generally different for different directions of motion and the apparent moment of inertia different for different axes of spins For this reason it seems better to speak of inertia coefficients, these being the constant coefficients in the expression for the kinetic energy in terms of the component linear and angular velocities relative to axes fixed in the body.

The idea of inertia coefficients may be extended to bodies of any shape and to cases in which there is more than one body or in which the fluid is limited by a boundary. Generalized coefficients may be defined, too, for cases in which there is circulation round some of the bodies or boundaries and values can eventually be obtained which should correspond closely to the values of the inertia coefficients for the motion of a body in a viscous fluid.

The inertia coefficients of airship hulls are useful for the interpretation of running tests and in fact for a dynamical study of any type of motion of an airship, whether steady or unsteady. The coefficients are needed, for instance, in the study of the stability of an airship by the method of small oscillations [1] and for a computation of the resulting momenta in various types of steady motion.

For the case of motion of translation with velocity U the kinetic energy, T, of the fluid is usually expressed in the form

$$T=\tfrac{1}{2}kmU^2$$

where m is the mass of the fluid displaced by the body and k is a numerical coefficient whose value is known in certain cases. A value of k for an airship hull is generally found by choos-

[1] For the literature on this subject reference may be made to a paper by R. Jones and D. H. Williams, British Aeronautical Research Committee, R. M. 751. June, 1921.

ing an ellipsoid with nearly the same form as the hull and calculating the value of k for the ellipsoid. This method is to some extent unsatisfactory because the coefficient k varies considerably with the shape, being infinite for a circular disk, 0.5 for a sphere, and 0.045 for a prolate spheroid of fineness ratio 6. For this reason an alternative method is proposed in which the kinetic energy of the fluid is expressed in terms of quantities relating to the master section of the hull by means of a formula involving a numerical coefficient K which varies only slowly with other characteristics such as the fineness ratio. The proposed expression is

$$T=\frac{4}{3}K\rho\frac{S^2U^2}{l}$$

where S denotes the area and l the perimeter of the greatest cross section of the hull by a plane perpendicular to the direction of motion; ρ is the density of the fluid and K the new coefficient which is apparently greatest for a circular or elliptic disk.

In the case of a spheroid moving in the direction of its axis of symmetry the way in which k and K vary with the fineness ratio is shown in Figure 1. In Figure 2 the corresponding curves have been drawn for a hull bounded by portions of two spheres cutting each other orthogonally. The high value of K when the two spheres are equal is undoubtedly caused by the presence of

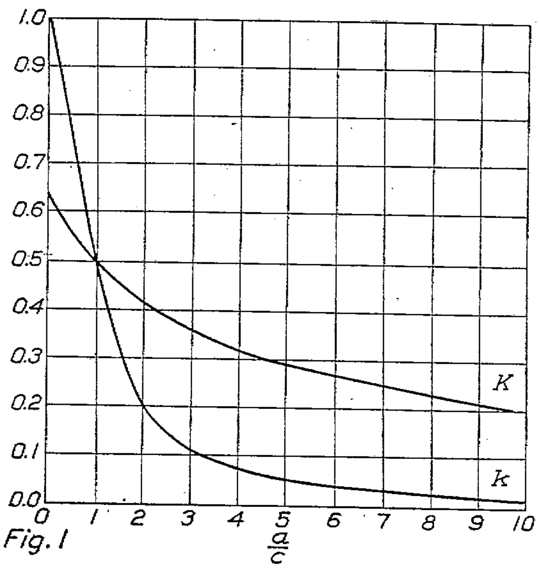

Fig. 1

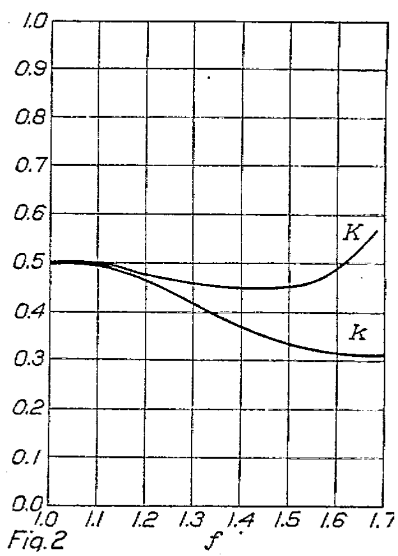

Fig. 2

the narrow waist, while the sudden drop in value indicates the effect of a lack of fore and aft symmetry. The curves for K have an advantage over those for k in indicating more clearly the effect of a change in shape. The effect of a flattening of the nose of the hull has been studied by considering the case of a surface of revolution whose meridian curve is a limaçon. The effect is only slight, as is seen from the table in Section IV. In the case of an airship hull spinning about a central axis in a plane of symmetry the kinetic energy can also be expressed in terms of general characteristics by using a formula involving a coefficient K, which varies only slowly with the shape. The proposed formula is

$$T=\frac{64}{45}K'\rho R_x^2\frac{(S_z-S_y)^2}{l}\omega_x^2$$

where ω_x is the angular velocity about the axis of spin, which we take as the axis of x, R_x is the maximum radius of gyration of a meridian section about the axis of x, S_y and S_z are the areas of central sections perpendicular to the axes of y and z and l is the perimeter of the meridian section with the greatest perimeter, a meridian section being cut out by a plane through the axis of spin.

This formula has been constructed from the known formula for an ellipsoid with the axes of coordinates as principal axes. To adapt it to a hull of a different shape a suitable set of axes must be chosen. The principal axes of inertia at the center of gravity may, perhaps, be used with advantage.

The coefficients k, K and K' will now be computed in some cases in which the aerodynamical problem is soluble. In particular they will be computed for the following cases:

(1) Disk moving axially.
(2) Prolate spheroid moving longitudinally.
(3) Prolate spheroid moving laterally.
(4) Oblate spheroid moving in the direction of its axis of symmetry.
(5) Oblate spheroid moving at right angles to its axis of symmetry.
(6) Solid formed by two orthogonal spheres.
(7) Solid formed by the revolution of a limaçon about is axis of symmetry.

II. THE INERTIA COEFFICIENTS FOR AN ELLIPSOID.

When the viscosity of the fluid is neglected and the motion is treated as irrotational there is no scale effect. This means that if we increase the velocity of the body in the ratio $s:1$, keeping its size constant, the velocity at any point of the fluid changes in the same proportion. A similar remark applies to the case in which the body is spinning about an axis instead of moving with a simple motion of translation and in the more general case in which a body has motions of both translation and rotation the kinetic energy, T, can be expressed in the form[2]

$$2T = Au^2 + Bv^2 + Cw^2 + 2A'vw + 2B'wu + 2C'uv + Pp^2 + Qq^2 + Rr^2 + 2P'qr + 2Q'rp + 2R'pq$$
$$+ 2p(Fu + Gv + Hw) + 2q(F'u + G'v + H'w) + 2r(F''u + G''v + H''w),$$

where (u, v, w) are the component velocities of a point fixed in the body and (p, q, r) are the angular velocities of the body about axes through this point that are likewise fixed in the body. The coefficients $A, B, C, A', B', C', P, Q, R, P', Q', R', F, G, H, F', F'', G', G'', H', H''$ are constants which are called the *inertia-coefficients* of the body relative to these axes. This expression for the kinetic energy has been used also in cases in which the velocities are variable and the determination of the inertia coefficients is evidently a matter of some importance.

The inertia coefficients are usually found by writing down the velocity potential or stream-line function which specifies the flow and calculating the kinetic energy by means of an integral of type

$$2T = -\rho \int \phi \frac{d\phi}{dn} dS$$

over the surface of the body, ϕ being the velocity potential, ρ the density of the fluid and dn denotes an element of the normal to the surface dS drawn into the fluid. A different integral may be used when the stream-line function is known, but in many cases integration is unnecessary, for Munk[3] has remarked that in the case of a simple velocity of translation the fluid motion may be supposed to arise from a series of doublets and that the sum of the moments of all these doublets has a component in the direction of motion which is proportional to the sum of the kinetic energy of the fluid and the kinetic energy which the fluid displaced would have if it moved like a rigid body with the same velocity as the body. The sum of the masses of the fluid and the fluid displaced has been called the *complete mass.*

The inertia-coefficients are well known for the case of an ellipsoid with semiaxes a, b, c when the axes of reference are the principal axes of the ellipsoid. We have in fact[2]

$$A = \frac{\alpha_0}{2 - \alpha_0} m, \quad P = \frac{(b^2 - c^2)^2 (\gamma_0 - \beta_0)}{2(b^2 - c^2) + (b^2 + c^2)(\beta_0 - \gamma_0)} \frac{m}{5}, \text{ etc.}$$

$$m = \frac{4}{3} \pi\rho\, abc, \quad A' = B' = C' = P' = Q' = R' = F = F' = F'' = G = G' = G'' = H = H' = H'' = o.$$

where

$$\alpha_0 = abc \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)\Delta}, \quad \beta_0 = abc \int_0^\infty \frac{d\lambda}{(b^2 + \lambda)\Delta}, \quad \gamma_0 = abc \int_0^\infty \frac{d\lambda}{(c^2 + \lambda)\Delta}, \quad \Delta = [(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)]^{\frac{1}{2}}.$$

[2] See Lamb's hydrodynamics.
[3] Technical Note No. 104, National Advisory Committee for Aeronautics. July, 1922.

The complete coefficients of inertia for motion of translation are

$$A^*=\frac{2m}{2-\alpha_0},\quad B^*=\frac{2m}{2-\beta_0},\quad C^*=\frac{2m}{2-\gamma_0}.$$

The coefficient k defined by the equation

$$k=\frac{\alpha_0}{2-\alpha_0}$$

has been tabulated by Professor Lamb in a number of cases.[4] We have extended his tables and have also tabulated the coefficient K defined in I. The different special cases of an ellipsoid will now be discussed.

1. *Elliptic disk.*—In this case k is infinite but the kinetic energy is finite. To find an expression for K we write

$$\alpha_0=\int_0^\infty\frac{abc\,d\lambda}{(a^2+\lambda)\Delta}=\sum_{n=0}^{\infty}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots2n}(c^2-b^2)^n\int_0^\infty\frac{abc\,d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{n+1}}$$

$$\int_0^\infty\frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)}=\frac{2}{\sqrt{c^2-a^2}}\left[\frac{\pi}{2}-\tan^{-1}\frac{a}{\sqrt{c^2-a^2}}\right]=\frac{\pi}{c}-\frac{2a}{c^2}+\frac{\pi}{2}\frac{a^2}{c^3}-\ .\ .\ .\ .$$

when a is small. Differentiating once with respect to a and n times with respect to c^2, we get

$$(-1)^n n!\int_0^\infty\frac{a\,d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(c^2+\lambda)^{n+1}}=\frac{2}{c^{2n+2}}(-1)^n n!-\frac{\pi a}{c^{2n+3}}(-1)^n\frac{3\cdot5\cdots(2n+1)}{2^n}+\ .\ .\ .$$

Hence

$$\alpha_0=\frac{2b}{c}\left[1+\frac{1}{2}\frac{c^2-b^2}{c^2}+\frac{1\cdot3}{2\cdot4}\left(\frac{c^2-b^2}{c^2}\right)^2+\ .\ .\ .\ .\right]-\frac{\pi ab}{c^2}\left[1+\frac{1\cdot3}{2^2}\frac{c^2-b^2}{c^2}+\frac{1\cdot3^2\cdot5}{2^2\cdot4^2}\left(\frac{c^2-b^2}{c^2}\right)^2+\ .\ .\ .\ .\right]$$

$$=2\left[1-abc\int_0^{\frac{\pi}{2}}\frac{d\theta}{(c^2\cos^2\theta+b^2\sin^2\theta)^{\frac{3}{2}}}+\text{higher powers of a}\right]=2\left[1-\frac{al}{4bc}\right]$$

approximately, where l is the perimeter of the ellipse with semiaxes b and c. Hence finally we obtain

$$2T=\frac{16}{3\pi}\frac{\rho\,U^2S^2}{l},\quad S=\pi bc$$

and[5]

$$K=\frac{2}{\pi}=0.637$$

The distribution of doublets may be found from the well-known expression for the potential We have for an ellipsoid

$$\phi=\frac{abc\,x\,U}{2-\alpha_0}\int_{\lambda_0}^\infty\frac{d\lambda}{(a^2+\lambda)\Delta},\quad\frac{x^2}{a^2+\lambda_0}+\frac{y^2}{b^2+\lambda_0}+\frac{z^2}{c^2+\lambda_0}=1$$

As $a\rightarrow O$ we have

$$\phi=\frac{2b^2c^2}{l}x\,U\int_{\lambda_0}^\infty\frac{d\lambda}{\lambda^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}},\quad\frac{x^2}{\lambda_0}+\frac{y^2}{b^2+\lambda_0}+\frac{z^2}{c^2+\lambda_0}=1$$

Putting $\lambda=x^2s$ and making $x\rightarrow O$ we find that the value of ϕ on the disk is

$$\phi_+=\frac{4bc}{l}U\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$$

[4] British Advisory Committee for Aeronautics, R. & M. No. 623, October (1918).

[5] In the case of an infinitely long strip bounded by two parallel lines the value of K is 0.589.

This must be equal to the 2π times the moment per unit area of the doublets in the neighborhood of the point (O, y, z) of the disk. Hence the expression for the potential is equivalent to

$$\phi=\frac{2bc}{\pi l}\,Ux\int\int\frac{\left[1-\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right]^{\frac{1}{2}}dy_0dz_0}{[x^2+(y-y_0)^2+(z-z_0)^2]^{\frac{3}{2}}}$$

and this formula shows the way in which the potential arises from the doublets. The complete energy is in this case

$$T=\frac{8}{3\pi}\frac{\rho U^2\pi^2b^2c^2}{l}=2\pi\rho U\left[\frac{2bc}{\pi l}U\int\int\left[1-\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right]^{\frac{1}{2}}dy_0dz_0\right.$$

in accordance with Munk's theorem. To verify this result we put

$$y_0=bs\cos\omega,\ z_0=cs\sin\omega$$

then

$$\int\int\left[1-\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right]^{\frac{1}{2}}dy_0dz_0=bc\int_0^1 s\sqrt{1-s^2}ds\int_0^{2\pi}d\omega=\frac{2\pi}{3}bc.$$

With the above substitution the expression for the potential may be written in the form

$$\phi=\frac{2bc}{\pi l}\,Ux\int^1 s\sqrt{1-s^2}ds\int_0^{2\pi}\frac{d\omega}{R^3},$$

$$R^2=x^2+(y-bs\cos\omega)^2+(z-cs\sin\omega)^2$$

and may be compared with the corresponding expression for the oblate spheroid. For the case of the circular disk $(b=c)$ the stream-line function may be obtained by replacing x in the above formula by $-\ (y^2+z^2)$. When an elliptic disk spins about the axis of y the kinetic energy is given by

$$2T=\frac{4}{15}\pi\rho c^2\Omega_y^2\div\int_0^{\frac{\pi}{2}}\frac{(1+\cos^2\theta)d\theta}{(c^2\cos^2\theta+b^2\sin^2\theta)^{\frac{3}{2}}}$$

where Ω_y is the angular velocity. In the case of a circular disk the kinetic energy is

$$\frac{8}{45}\rho c^5\Omega_y^2$$

The coefficient K^1 thus has the value

$$K^1=\frac{1}{\pi}=0.318.$$

2. *Prolate spheroid.*—In the case of a prolate spheroid moving in the direction of its axis of symmetry, we have (Lamb, loc. cit.)

$$\alpha_0=\frac{2(1-e^2)}{e^3}\left\{\frac{1}{2}\log\frac{1+e}{1-e}-e\right\}$$

where e is the eccentricity of the meridian section and so

$$b=c=a\sqrt{1-e^2}$$

The velocity potential is

$$\phi_a=\frac{ab^2Ux}{2-\alpha_0}\int_\lambda^\infty\frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)}$$

where

$$\frac{x^2}{a^2+\lambda}+\frac{y^2+z^2}{b^2+\lambda}=1$$

By introducing the spheroidal coordinates

$$x=h\ \mu\zeta,\ y=\omega \cos w,\ z=\omega \sin w,\ \omega=h\ (1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}},\ h=ae$$

we may write this in the form

$$\phi_a=A\ P_1(\mu)\ Q_1(\zeta)=A\ \mu\left\{\frac{1}{2}\zeta\ log\frac{\zeta+1}{\zeta-1}-1\right\}$$

where

$$A\left\{\frac{1}{1-e^2}-\frac{1}{2e}\log\frac{1+e}{1-e}\right\}=a\ U.$$

The velocity potential may also be expressed as a definite integral

$$\phi_a=\frac{1}{2}\ Ah\int_{-1}^{+1}\frac{sds}{[\ (x-hs)^2+y^2+z^2]^{\frac{1}{2}}}$$

which indicates the way in which it may be imagined to arise from a row of sources and sinks on the line joining the foci. This result may be obtained by writing

$$\mu\left\{\frac{1}{2}\zeta\ \log\frac{\zeta+1}{\zeta-1}-1\right\}=\frac{1}{2}h\int_{-1}^{+1}\frac{f(s)ds}{[(x-hs)^2+y^2+z^2]^{\frac{1}{2}}}$$

and determining $f(s)$ from the integral equation

$$Q_1(\zeta)=\frac{1}{2}h\int_{-1}^{+1}\frac{f(s)ds}{h(\zeta-s)}$$

which is obtained by putting $y=z=0$. The integral equation is solved most conveniently by using the well-known expansion

$$\frac{1}{\zeta-s}=\sum_0^\infty\ (2n+1)\ P_n(s)Q_n(\zeta)$$

and the integral formula

$$\int_{-1}^{+1}P_m(s)P_n(s)ds=\begin{matrix}0 & m\pm n\\ \frac{2}{2\ n+1} & m=n\end{matrix}$$

It is thus evident that

$$f(s)=P_1(s)=s.$$

The strength of the source associated with an element hds is

$$\frac{1}{2}\ Ah.\ sds.\ 4\pi\rho$$

Multiplying this by $Ux=Uhs$ and integrating with regard to s between -1 and 1, we get

$$\frac{4\pi}{3}Ah^2\rho\ U=\frac{4\pi\rho}{3}\frac{ah^2\ U^2}{\frac{1}{1-e^2}-\frac{1}{2\ e}\log\frac{1+e}{1-e}}$$

The kinetic energy of the fluid plus the kinetic energy of the fluid displaced is, on the other hand

$$\frac{4\pi\rho}{3}ac^2.\frac{1}{2}\ U^2\left[1+\frac{\alpha_0}{2-\alpha_0}\right]$$

and

$$2-\alpha_0=\frac{2(1-e^2)}{e^2}\left[\frac{1}{1-e^2}-\frac{1}{2e}\log\frac{1+e}{1-e}\right]$$

Thus Munk's theorem is again confirmed.

In the case of a prolate spheroid moving broadside on we have

$$\beta_0=\frac{1}{e^2}-\frac{1-e^2}{2\,e^3}\log\frac{1+e}{1-e}$$

and the relation between K and k is

$$K=\frac{lk}{2\pi a}$$

where l is the perimeter of the meridian section. The potential ϕ_b may be expressed in the forms

$$\phi_b=\frac{ab^2Vy}{2-\beta_0}\int_\lambda^\infty\frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^2}$$

where

$$\frac{x^2}{a^2+\lambda}+\frac{y^2+z^2}{b^2+\lambda}=1$$

$$\phi_b=A(1-\mu^2)^{\frac{1}{2}}\,(\zeta^2-1)^{\frac{1}{2}}\left\{\frac{1}{2}\log\frac{\zeta+1}{\zeta-1}-\frac{\zeta}{\zeta^2-1}\right\}\cos w$$

where

$$A\left\{\frac{1}{2}\log\frac{1+e}{1-e}-\frac{e(1-2\,e^2)}{1-e^2}\right\}=-hV$$

$$\phi_b=-\frac{1}{2}\;Ayh\int_{-1}^{+1}\frac{(1-s^2)\,ds}{[(x-hs)^2+y^2+z^2]^{\frac{3}{2}}}$$

At Doctor Munk's suggestion one may interpret these results with the aid of the idea of complete momentum, i. e., the momentum of the fluid plus the momentum which the fluid displaced by the body would have if it moved like a rigid body with the same velocity as the body.

Let M_a and M_b denote the complete masses for motions parallel to the axes of x and y respectively, then

$$M_a=\frac{4}{3}\,\pi\rho abc\;(1+k_a)\quad M_b=\frac{4}{3}\,\pi\rho abc\;(1+k_b)$$

and we may write

$$\Phi_a=\frac{3\,M_a\,U}{8\pi h}\int_{-1}^{+1}\frac{sds}{[(x-hs)^2+y^2+z^2]^{\frac{3}{2}}}$$

$$\Phi_b=\frac{3M_b\,Vy}{16\pi}\int_{-1}^{+1}\frac{(1-s^2)\;ds}{[\,(x-hs)^2+y^2+z^2]^{\frac{3}{2}}}$$

These equations show that when the complete momentum is given the velocity potential Φ and the sources from which it arises are the same for a series of confocal spheroids.[6] This is true for any angle of attack as is seen by superposition. This result is easily extended to the ellipsoid, for we may write

$$\Phi_a=\frac{3}{8\pi}M_a\,Ux\Gamma,$$

where

$$\Gamma=\int_{\lambda_0}^\infty\frac{d\lambda}{(a^2+\lambda)\;\Delta},$$

$$\frac{x^2}{a^2+\lambda_0}+\frac{y^2}{b^2+\lambda_0}+\frac{z^2}{c^2+\lambda_0}=1.$$

[6] This is an extension to three dimensions of a theorem that has been proved for the elliptic cylinder. Cf. Max. M. Munk, Notes on Aerodynamic Forces. Technical Note No. 104, National Advisory Committee for Aeronautics.

It is easily seen that Γ is the same for a system of confocal ellipsoids. This result may be used to find an appropriate system of singularities distributed over the region bounded by a real confocal ellipse, the result is the same as that already found for the elliptic disk.[7]

It is well known that an ellipsoid has three focal conics, one of which is imaginary, and the question arises whether there is more than one simple distribution of singularities which will produce the potential. This question will be discussed in Section III.

When a prolate spheroid is spinning with angular velocity Ω_y about the axis of y, the velocity potential Φ is given by the formulæ

$$\Phi = A\mu \ (1-\mu^2)^{\frac{1}{2}} \ (\zeta^2-1)^{\frac{1}{2}} \left\{\frac{3}{2}\zeta \log\frac{\zeta+1}{\zeta-1} - 3 - \frac{1}{\zeta^2-1}\right\} \text{Sin } w = -\frac{1}{2} Az \int_{-1}^{+1} \frac{s\ (1-s^2)\ ds}{[(x-hs)^2+y^2+z^2]^{\frac{3}{2}}}$$

where A is a constant to be determined by means of the boundary condition

$$\frac{\delta\Phi}{\delta\zeta} = -\Omega_y\left(z\frac{\delta x}{\delta\zeta} - x\frac{\delta z}{\delta\zeta}\right)$$

It is easily seen that

$$A\left[\frac{3}{2e^2}(2-e^2)\log\frac{1+e}{1-e} - \frac{8}{e} - \frac{e}{1-e^2}\right] = a^2e^2\Omega_y$$

The energy may be expressed in terms of the mass of the fluid displaced by means of the formula

$$2T = k^1 m\left(\frac{a^2}{5} - \frac{c^2}{5}\right)\Omega^2_y$$

(the coefficient k^1 having been tabulated by Lamb) or it may be expressed in terms of other characteristics with the aid of our coefficient K^1. The values of the various coefficients k and K are given in Table I. The suffixes a and c are used to indicate the axis along which the spheriod is moving. The coefficients k^1 and K^1 refer to the case of rotation. It will be seen that the coefficients K vary only slowly and the same remark applies to the product $(1+k_a)(1+k_c)$. One advantage in using the coefficients K_a and K_c is that it is not necessary to compute the volume of the hull of the airship. Since K_c varies very slowly indeed when the fineness ratio a/c is in the neighborhood of 6, it follows that if we take $K_c = 0.6$ for an airship hull we shall not be far wrong.

TABLE I.

$\frac{a}{c}$	k_a	k_c	$\frac{(1+k_a)}{(1+k_c)}$	K_a	K_c	k'	K'
1.00	0.500	0.500	2.250	0.500	0.500	0	1
1.155	....	....	....	....	....	....	0.906
1.50	0.305	0.621	2.116	0.457	0.523	0.244	....
2.00	0.209	0.702	2.058	0.418	0.541	0.399	0.695
2.065	....	....	....	....	....	....	0.685
2.99	0.122	0.803	2.023	0.365	0.571	0.582	....
3.571							0.512
3.99							....
4.99							....
6.01	0.045	0.918	2.004	0.270	0.606	0.807	....
6.97	0.036	....	....	0.250	....	....	....
8.01	0.029	....	....	0.232	....	....	....
9.02	0.024	....	....	0.216	....	....	....
9.97	0.021	....	....	0.209	....	....	....
∞	0	1	2	0	0.637	1	0.477

In this table use has been made of the coefficients computed by Lamb. It should be noticed that $K_a + 2K_c$ is very nearly constant for values of $\frac{a}{c}$ lying between 1 and 6. This fact may be used to compute K_c when K_a is known using a formula such as

$$K_c = \cdot 743 - \frac{1}{2}K_a$$

The value thus found is too large for large values of $\frac{a}{c}$ and too small for small values of $\frac{a}{c}$.

[7] Cf. Lamb's Hydrodynamics, 3d ed., ch. V, p. 145.

3. *Oblate spheroid.*—In the case of an oblate spheroid moving with velocity U in the direction of its axis of symmetry, which we take as axis of x, we have

$$\alpha_0=\frac{2}{e^2}\left[1-\frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right]$$

where e is the eccentricity of the meridian section. In this case

$$b=c,\ a=c\sqrt{1-e^2}.$$

The velocity potential is

$$\phi_a=\frac{ac^2Ux}{2-\alpha_0}\int_{\lambda_0}^{\infty}\frac{d\lambda}{a^2+\lambda)^{\frac{3}{2}}(c^2+\lambda)}$$

where

$$\frac{x^2}{a^2+\lambda_0}+\frac{y^2+z^2}{c^2+\lambda_0}=1.$$

Introducing the spheroidal coordinates

$$x=h\mu\zeta,\ y=\omega\cos w,\ z=\omega\sin w,\ \omega=h\ (1-\mu^2)^{\frac{1}{2}}\ (\zeta^2+1)^{\frac{1}{2}},\ h=ce.$$

we may write

$$\phi_a=A\mu\ (1-\zeta\cot^{-1}\zeta)$$

where

$$A\left\{\frac{a\sqrt{c^2-a^2}}{c^2}-\cos^{-1}\frac{a}{c}\right\}=-h^3U.$$

We also have

$$\phi_a=\frac{Ax}{2\pi}\int_0^1 s\sqrt{1-s^2}\,ds\int_0^{2\pi}\frac{dw}{R^3}$$

$$R^2=(y-hs\cos w)^2+(z-hs\sin w)^2+x^2.$$

When an oblate spheroid moves with velocity W at right angles to its axis of symmetry we have

$$\beta_0=\gamma_0=\frac{\sqrt{1-e^2}}{e^3}\left[\sin^{-1}e-e\sqrt{1-e^2}\right]$$

and the relation between k_c and K_c is now

$$K_c=\frac{lk_c}{2\pi a}.$$

The velocity potential ϕ_c is given by the formulæ

$$\phi_c=\frac{ac^2Wz}{2-\gamma_0}\int_{\lambda_0}^{\infty}\frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^2}$$

$$=A(1-\mu^2)^{\frac{1}{2}}\ (\zeta^2+1)^{\frac{1}{2}}\left\{\frac{\zeta}{\zeta^2+1}-\cot^{-1}\zeta\right\}\sin w$$

$$=\frac{A}{2\pi}\int_0^1 2s\sqrt{1-s^2}\,ds\int_0^{2\pi}\frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

$$R^2=(y-hs\cos w)^2+(z-hs\sin w)^2+x^2,\quad h=ce=\sqrt{c^2-a^2}$$

$$A\left\{\cos^{-1}\frac{a}{c}-\frac{a^2+2c^2}{ac^2}\sqrt{c^2-a^2}\right\}=h^3W.$$

Some values of the coefficients k_a, k_c, K_a, K_c, are given in Table II.

TABLE II.

$\frac{c}{a}$	k_a	K_a	k_c	K_c	$(1+k_a)(1+k_c)$
1.00	0.500	0.500	0.500	0.500	2.250
1.50	0.621	0.523	0.382	0.4824	2.240
2.00	0.702	0.541	0.310	0.477	2.229
2.99	0.803	0.571	0.223	0.473	2.205
3.99	0.860	0.587	0.174	0.474	2.184
4.99	0.895	0.599	0.143	0.477	2.166
6.01	0.918	0.606	0.1205	0.478	2.149
∞	1.000	0.637	0.000	0.500	2.000

When an oblate spheroid spins with angular velocity ω_y about the axis of y the velocity potential ϕ' is given by the formulæ

$$\phi = Czx\int_{\lambda_0}^{\infty}\frac{d\lambda}{(c^2+\lambda)(a^2+\lambda)\Delta}$$

$$C=\frac{(c^2-a^2)^2abc\omega_y}{2(c^2-a^2)+(c^2+a^2)(\gamma_0-\alpha_0)}$$

$$\phi' = A\mu\,(1-\mu^2)^{\frac{1}{2}}\,(\zeta^2+1)^{\frac{1}{2}}\left\{3\zeta\cot^{-1}\zeta-3+\frac{1}{\zeta^2+1}\right\}$$

$$A\left\{3\frac{c^2+a^2}{c^2-a^2}\cos^{-1}\frac{a}{c}-\frac{a}{c^2}\,\frac{7c^2+a^2}{\sqrt{c^2-a^2}}\right\}=h^5\omega_y.$$

We also have

$$\phi' = -\frac{A}{3\pi}\int_0 s\,(1-s^2)^{\frac{1}{2}}\,ds\int_0^{2\pi}\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{R}\right)dw$$

III. THE METHOD BASED ON THE USE OF SOURCES AND SINKS.

It was shown by Stokes[8] that the velocity potential for the irrotational motion of an incompressible nonviscous fluid in the space outside a sphere of radius, a, moving with velocity U, is the same as that of a doublet of moment $2\pi Ua^3$ situated at the center of the sphere. This result has been generalized by Rankine,[9] D. W. Taylor,[10] Fuhrmann,[11] Munk,[12] and others, two sources of opposite signs at a finite distance apart giving stream lines shaped like an airship.

Munk has shown in a recent report that the intensity of the point source near one end of an airship hull may be taken to be $r^2\pi U$, where r is the radius of the greatest section of the ship and $\frac{1}{2}r$ the distance of the point source from the head of the ship. The total energy of the fluid displaced is then

$$T=\frac{1}{6}\pi r^3\rho U^2$$

and the apparent increment of mass of the airship is equivalent to about $2\frac{1}{2}$ per cent of the mass of fluid displaced.

In this investigation the airship is treated as symmetrical fore and aft, the two sources of opposite signs being equidistant from the two ends and the contributions of the two sources to the kinetic energy being equal. The final result is identical with that for an elongated spheroid with a ratio of axes equal to 9.

[8] Cambr. Phil Trans., vol. 8 (1843). [Math. and Phys. Papers, Vol. I. p. 17.]
[9] Phil. Trans. London (1871), p. 267.
[10] Trans. British Inst. Naval Architects, vol. 35 (1894), p. 385.
[11] Jahrb. der Motorluftschiff-Studiengesellschaft, 1911–12.
[12] National Advisory Committee for Aeronautics, Reports 114 and 117 (1921).

It is thought that a lack of fore and aft symmetry will still further reduce the values of the coefficients k and K. To get an idea of the effect of a lack of symmetry we shall consider the case of a solid bounded by portions of two orthogonal spheres. In this case, as is well known, the velocity potential may be derived from three collinear sources. We may in fact write

$$\phi=\frac{1}{2}a^3\,U\,\frac{\cos\theta}{r^2}+\frac{1}{2}a'^3\,U\,\frac{\cos\theta'}{r'^2}-\frac{1}{2}p^3\,U\,\frac{\cos\Theta}{R^2}$$

$$\psi=\frac{1}{2}a^3\,U\,\frac{\sin^2\theta}{r}+\frac{1}{2}a'^3\,U\,\frac{\sin^2\theta'}{r'}-\frac{1}{2}p^3\,U\,\frac{\sin^2\Theta}{R}$$

where a and a' are the radii of the two spheres (r, θ), $(r'\ \theta')$, (R, Θ) are polar coordinates referred to the three sources as poles, the angles being measured from the line joining the three sources. If Q is a common point of the two spheres R is measured from the foot of the perpendicular from Q on the line of centers, while r and r' are measured from the centers of the two spheres respectively. The quantity p represents the distance of Q from the line of centers and is given by the equation

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{a'^2}.$$

By means of Munk's theorem we infer that the complete energy is given by the formula

$$2\,T=2\pi\rho\,[a^3+a'^3-p^3]\,U^2=\rho\,(1+k)\,VU^2$$

where

$$V=\frac{\pi}{3}\left[2\,(a^2+a'^2)^{\frac{3}{2}}+2a^3+2a'^3-3\,\frac{a^2a'^2}{(a^2+a'^2)^{\frac{1}{2}}}\right]$$

is the volume of the fluid displaced.

The fineness ratio, i. e., the ratio of the length to the greatest breadth, is

$$f=\frac{a+a'+\sqrt{a^2+a'^2}}{2a}.$$

Some values of k and K are given in Table III and curves have been drawn in Figure 2 to show the effect of a lack of fore and aft symmetry. For a comparison we have given in Table III the values of k and K for a spheroid of the same fineness ratio. The high value of K for the two orthogonal spheres is undoubtedly due to the presence of a narrow waist. The sudden drop in the value of K is probably due to the lack of fore and aft symmetry. The coefficient K shows the effect of a change in shape much more clearly than the coefficient k.

TABLE III.

$\frac{a^1}{a}$	k	K	f	k (spheroid).	K(spheroid).
1.0	0.313	0.5897	1.707	0.243	0.440
0.9	0.315	0.5136	1.622		
0.8	0.329	0.4708	1.54		
0.75	0.334	0.4509	1.5	0.305	0.457
0.66	0.363	0.4603	1.434		
0.41	0.448	0.471	1.25		
0.29	0.48	0.488	1.166		
0	0.5	0.5	1	0.5	0.5

It appears from an examination of the case of the oblate spheroid that the motion of air round a moving surface of revolution can not always be derived from a number of sources at real points on the axis. For the oblate spheroid the sources, or rather doublets, are in the equitorial plane. It is possible, however, to replace these doublets by doublets at imaginary points on the axis as the following analysis will show.

If $F(x, y, z)$ is a potential function, we have the equation [13]

$$\frac{1}{2\pi}\int_0^{2\pi} F[x, y-\sigma \cos \omega, x-\sigma \sin \omega]\, d\omega = \frac{1}{\pi}\int_0^{\pi} F[x+i\sigma \cos X, y, z]\, dX$$

which holds under fairly general conditions. On account of this equation we may write

$$\frac{1}{2\pi}\int_0^{h} f(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial}{\partial x}\left(\frac{1}{R}\right) d\omega = \frac{1}{\pi}\int_0^{h} f(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial}{\partial x}\left(\frac{1}{R'}\right) dX$$

$$\frac{1}{2\pi}\int_0^{h} f_1(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial}{\partial z}\left(\frac{1}{R}\right) d\omega = \frac{1}{\pi}\int_0^{h} f_1(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial}{\partial z}\left(\frac{1}{R'}\right) dX$$

$$\frac{1}{2\pi}\int_0^{h} g(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial^2}{\partial z \partial x}\left(\frac{1}{R}\right) d\omega = \frac{1}{\pi}\int_0^{h} g(\sigma)\, d\sigma \int_0^{2\pi} \frac{\partial^2}{\partial z \partial x}\left(\frac{1}{R'}\right) dX$$

where

$$R^2 = x^2 + (y-\sigma \cos \omega)^2 + (z-\sigma \sin \omega)^2,\quad R'^2 = (x+i\sigma \cos X)^2 + y^2 + z^2.$$

Putting

$$f(\sigma) = f_1(\sigma) = \sigma\,(h^2-\sigma^2)^{\frac{1}{2}},\quad g(\sigma) = \sigma\,(h^2-\sigma^2)^{\frac{3}{2}}$$

making the substitution

$$\sigma \cos X = \xi, \quad dX\sqrt{\sigma^2-\xi^2} = -d\xi,$$

changing the order of integration and making use of the equations

$$\int_\xi^h d\sigma\, \frac{\sigma\sqrt{h^2-\sigma^2}}{\sqrt{\sigma^2-\xi^2}} = \frac{\pi}{4}\,(h^2-\xi^2),$$

$$\int_\xi^h d\sigma\, \frac{\sigma(h^2-\sigma^2)^{\frac{3}{2}}}{\sqrt{\sigma^2-\xi^2}} = \frac{3\pi}{16}\,(h^2-\xi^2)^2,$$

which are easily verified by means of the substitution

$$\sigma^2 = \xi^2 \cos^2 \theta + h^2 \sin^2 \theta,$$

we find that the potentials for the oblate spheroid in the three types of motion may be written in the forms

$$\phi_a = -\frac{A}{4h^3}\int_{-h}^{h} (h^2-\xi^2)\, d\xi\, \frac{\partial}{\partial x}\left(\frac{1}{R''}\right) = \frac{iA}{2h^3}\int_{-h}^{h} \frac{\xi d\xi}{R''}$$

$$\phi_c = -\frac{A}{2h^3}\int_{-h}^{h} (h^2-\xi^2)\, d\xi\, \frac{\partial}{\partial z}\left(\frac{1}{R''}\right)$$

$$\phi' = -\frac{A}{8h^5}\int_{-h}^{h} (h^2-\xi^2)^2 d\xi\, \frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R''}\right) = +\frac{iA}{2h^5}\int_{-h}^{h} \xi\,(h^2-\xi^2)\, d\xi\, \frac{\partial}{\partial z}\left(\frac{1}{R''}\right)$$

where

$$R'' = [(x+i\xi)^2 + y^2 + z^2]^{\frac{1}{2}}.$$

These formulæ resemble those for the prolate spheroid.

A distribution of sources or doublets over the elliptic area bounded by a focal ellipse of an ellipsoid may be replaced by a system of sources or doublets at imaginary points in one of the other planes of symmetry by making use of the equation.[14]

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[13] H. Bateman, Amer. Journ. of Mathematics, vol. 34 (1912), p. 335.
[14] H. Bateman, loc. cit., p. 336.

$$\int_0^{2\pi} F[x-\sigma\cos\theta\cos\alpha,\, y-\sigma\sin\theta,\, z]\,d\theta=\int_0^{2\pi} F[x+i\sigma\sin X\sin\alpha,\, y,\, z+i\sigma\cos X]\,dX$$

which likewise holds under fairly general conditions when $F(x, y, z)$ is a potential function and α an arbitrary constant.

The theorem relating to the transformation of doublets in a central plane into a series of doublets at imaginary points on the axis of symmetry may be written in the general form

$$\frac{1}{2\pi}\int_0^{h} f(\sigma)\,d\sigma\int_0^{2\pi} G\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{R}\,d\omega=\frac{1}{\pi}\int_{-h}^{h} G\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{R'}\,F(\xi)\,d\xi$$

where the functions $f(\sigma)$ and $F(\xi)$ are connected by the integral equation

$$F(\xi)=\int_\xi^{h}\frac{f(\sigma)\,d\sigma}{\sqrt{\sigma^2-\xi^2}}.$$

In order that Munk's theorem may be applicable to doublets at imaginary points as wel as to doublets at real points we must have the equation

$$\frac{1}{\pi}\int_{-h}^{h} F(\xi)\,d\xi=\int_0^{h} f(\sigma)\,d\sigma.$$

Now

$$\frac{1}{\pi}\int_{-h}^{h} d\xi\int_\xi^{h}\frac{f(\sigma)\,d\sigma}{\sqrt{\sigma^2-\xi^2}}=\frac{1}{\pi}\int_0^{h} f(\sigma)\,d\sigma\int_{-\sigma}^{\sigma}\frac{d\xi}{\sqrt{\sigma^2-\xi^2}}=\int_0^{h} f(\sigma)\,d\sigma$$

hence the formula is verified and the complete mass may be calculated from doublets at imaginary points by adding the moments and using Munk's formula.

IV. CASES IN WHICH THE MASS CAN BE FOUND WITH THE AID OF SPECIAL HARMONIC FUNCTIONS

It is known that the potential problem may be solved in certain cases by using series of spheroidal, toroidal, bipolar, or cylindrical harmonics. Thus it may be solved for the spherical bowl, anchor ring, two spheres,[15] and for the body formed by the revolution of a limaçon about its axis of symmetry. The last case is of some interest, as it indicates the effect of a flattening of the nose of an airship hull. Writing the equation of the limaçon in the form

$$r=2a^2\,\frac{s+\cos\theta}{s^2-1}$$

where r and θ are polar coordinates, we find on making the substitutions

$$r\cos\theta=x=\xi^2-\eta^2.\quad r\sin\theta=y=2\xi\eta$$

$$\xi=\frac{a\sinh\sigma}{\cosh\sigma-\cos\chi}.\quad \eta=\frac{a\sin\chi}{\cosh\sigma-\cos\chi}$$

that the potential for motion parallel to the axis of symmetry is

$$\phi=\frac{1}{2}a^2U\sum_{m=0}^{\infty}(m+1)\left[(m+2)^2\frac{Q'_{m+1}(s)}{P'_{m+1}(s)}-m^2\frac{Q'_m(s)}{P'_m(s)}\right]$$

$$[P_m(\cosh\sigma)\,P_{m+1}(\cos\chi)-P_{m+1}(\cosh\sigma)\,P_m(\cos\chi)]$$

[15] For references see Lamb's Hydrodynamics, 3d ed., pp. 126, 149; and A. B. Basset, Hydrodynamics, Cambridge, 1888, Vol. I.

where $P_m(s)$ and $Q_m(s)$ are the two types of Legendre functions (zonal harmonics) and $P'_m(s)$, $Q'_m(s)$ are the derivatives of $P_m(s)$ and $Q_m(s)$ respectively. The stream-line function ψ as found by Basset is in our notation.

$$\psi = -\frac{a^4 U}{\cosh\sigma - \cos\chi}\sum_{m=0}^{\infty}(2m+3)\frac{Q'_{m+1}(s)}{P'_{m+1}(s)}P'_{m+1}(\cosh\sigma)\,P'_{m+1}(\cos\chi)\sinh^2\sigma\sin^2\chi$$

At a great distance from the origin we have the approximate expressions

$$\phi = -\frac{1}{2}a^6\,U\frac{x}{r^3}S,\ \psi = -\frac{1}{2}a^6\,U\frac{y^2}{R^3},\ \frac{2a^2}{R} = \cosh\sigma - \cos\chi,$$

$$S = \sum_{m=0}^{\infty}(2m+3)\ (m+1)^2\ (m+2)^2\frac{Q'_{m+1}(s)}{P'_{m+1}(s)}$$

which give the sum of the moments of the doublets from which the potential arises. The coefficients k amd K may now be calculated with the aid of Doctor Munk's theorem and an incomplete table of spheroidal harmonics which is in the author's possession. We thus obtain the values [16]

TABLE IV.

s	f	k	K	k (spheroid)	K (spheroid)
∞	1	0.500	0.500	0.500	0.500
3	1.05	0.527	0.507	0.512	0.502
2	1.10	0.548	0.513	0.524	0.505
1.2	1.153	0.569	0.518	0.536	0.507
1.1	1.154	0.573	0.523	0.536	0.507
1	1.155	0.578	0.527	0.536	0.507

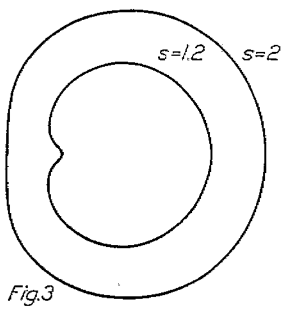

Fig.3

The corresponding values for an oblate spheroid are given for comparison. The case in which $s=2$ is particularly interesting because the limacon then has a point of undulation at the nose. When $s<2$ the limacon curves inward at the front, as may be seen from the diagrams in Figure 3, and the apparent mass is probably increased on account of fluid being confined in the hollow. In calculating the fineness ratio in such a case the length has been measured from the rear to the point where the double tangent meets the axis.

[16] The values for the spheroid have been obtained by interpolation from Table II. The values of k and K for the cardioid $s=1$ have been estimated by extrapolation.