
REPORT No. 184

**THE AERODYNAMIC FORCES ON
AIRSHIP HULLS**

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National Advisory Committee
for Aeronautics

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SUMMARY

This report describes the new method for making computations in connection with the study of rigid airships, which was used in the investigation of Navy's *ZR-1* by the special subcommittee of the National Advisory Committee for Aeronautics appointed for this purpose. It presents the general theory of the air forces on airship hulls of the type mentioned, and an attempt has been made to develop the results from the very fundamentals of mechanics, without reference to some of the modern highly developed conceptions, which may not yet be thoroughly known to a reader uninitiated into modern aerodynamics, and which may perhaps for all times remain restricted to a small number of specialists.

I. GENERAL PROPERTIES OF AERODYNAMIC FLOWS.

The student of the motion of solids in air will find advantage in first neglecting the viscosity and compressibility of the latter. The influence of these two properties of air are better studied after the student has become thoroughly familiar with the simplified problem. The results are then to be corrected and modified; but in most cases they remain substantially valid.

Accordingly I begin with the discussion of the general properties of aerodynamic flows produced by the motion of one or more solid bodies within a perfect fluid otherwise at rest. In order to be able to apply the general laws of mechanics to fluid motion I suppose the air to be divided into particles so small that the differences of velocity at different points of one particle can be neglected. This is always possible, as sudden changes of velocity do not occur in actual flows nor in the kind of flows dealt with at present. The term "flow" denotes the entire distribution of velocity in each case.

With aerodynamic flows external volume forces (that is, forces uniformly distributed over the volume) do not occur. The only force of this character which could be supposed to influence the flow is gravity. It is neutralized by the decrease of pressure with increasing altitude, and both gravity and pressure decrease can be omitted without injury to the result. This does not refer to aerostatic forces such as the buoyancy of an airship, but the aerostatic forces are not a subject of this paper.

The only force acting on a particle is therefore the resultant of the forces exerted by the adjacent particles. As the fluid is supposed to be nonviscous, it can not transfer tensions or forces other than at right angles to the surface through which the transfer takes place. The consideration of the equilibrium of a small tetrahedron shows, then, that the only kind of tension possible in a perfect fluid is a pressure of equal magnitude in all directions at the point considered.

In general this pressure is a steady function of the time t and of the three coordinates of the space, say x , y , and z , at right angles to each other. Consider now a very small cube with the edges dx , dy , and dz . The mean pressure acting on the face $dy\ dz$ may be p . The mean pressure on the opposite face is then $p + \partial p / \partial x dx$. The X -component of the resultant volume force is the difference of these two mean pressures, multiplied by the area of the faces $dydz$,

hence, it is $-\frac{\partial p}{\partial x} dx dy dz$. Per unit volume it is $-\frac{\partial p}{\partial x}$, as the volume of the cube is dx, dy, dz . It can be shown in the same way that the other two components of the force per unit volume are $-\frac{\partial p}{\partial y}$ and $-\frac{\partial p}{\partial z}$. Such a relation as existing between the pressure distribution and the force produced by it is generally described as the force being the "gradient" of the pressure, or rather the negative gradient. Any steady distribution of pressure has a gradient at each point, but if a distribution of forces (or of other vectors) is given, it is not always possible to assign a quantity such that the forces are its gradient.

We denote the density of air by ρ ; that is, the mass per unit volume, assumed to be constant. $d\tau$ may denote the small volume of a particle of air. The mass of this particle is then $\rho d\tau$. The components of the velocity V of this particle parallel to x, y , and z may be denoted by u, v , and w . Each particle has then the kinetic energy $dT = \frac{\rho}{2} d\tau (u^2 + v^2 + w^2)$ and the component of momentum, say in the X direction, is $\rho d\tau u$. The kinetic energy of the entire flow is the integral of that of all particles.

$$T = \frac{\rho}{2} \int (u^2 + v^2 + w^2) d\tau \dots\dots\dots (1)$$

Similarly, the component of momentum in the X -direction is the integral

$$\rho \int u d\tau \dots\dots\dots (2)$$

and two similar equations give the components for the two other directions. These integrals will later be transformed to make them fit for actual computation of the energy and the momentum.

It is sometimes useful to consider very large forces, pressures, or volume forces acting during a time element dt so that their product by this time element becomes finite. Such actions are called "impulsive." Multiplied by the time element they are called impulses, or density of impulse per unit area or unit volume as the case may be.

After these general definitions and explanations, I proceed to establish the equations which govern an aerodynamic flow. Due to the assumed constant density, we have the well-known equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \dots\dots\dots (3)$$

We turn now to the fact that for aerodynamic problems the flow can be assumed to be produced by the motion of bodies in air originally at rest. As explained above, the only force per unit volume acting on each particle is the gradient of the pressure. Now, this gradient can only be formed and expressed if the pressure is given as a function of the space coordinates x, y , and z . The laws of mechanics, on the other hand, deal with one particular particle, and this does not stand still but changes its space coordinates continually. In order to avoid difficulties arising therefrom, it is convenient first to consider the flow during a very short time interval dt only, during which the changes of the space coordinates of the particles can be neglected as all velocities are finite. The forces and pressures, however, are supposed to be impulsive, so that during the short interval finite changes of velocity take place. Suppose first the fluid and the bodies immersed therein to be at rest. During the creation of the flow the density of impulse per unit area may be P , i. e., $P = \int p dt$. The principles of mechanics give then

$$u\rho = -\frac{\partial P}{\partial x}$$

$$u = \frac{\partial}{\partial x} \left(-\frac{P}{\rho} \right)$$

and similarly in the two other directions

$$v = \frac{\partial}{\partial y} \left(-\frac{P}{\rho} \right) \dots\dots\dots (4)$$

$$w = \frac{\partial}{\partial z} \left(-\frac{P}{\rho} \right)$$

Hence the velocity thus created is the gradient of $\left(-\frac{P}{\rho} \right)$. At this state of investigation the value of $\frac{P}{\rho}$ is not yet known. But the important result is that the flow thus created is of the type having a distribution of velocity which is a gradient of some quantity, called the velocity potential Φ . Φ is the impulse density which would stop the flow, divided by the density ρ . According to (4)

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z} \dots\dots\dots (5)$$

from which follows

$$\Phi = \int (u dx + v dy + w dz) \dots\dots\dots (6)$$

A second differentiation of (5) gives

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \text{ etc.} \dots\dots\dots (7)$$

since both are equal to $\frac{\partial^2 \Phi}{\partial x \partial y}$. The substitution of (5) into the equation of continuity (3) gives

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \dots\dots\dots (8)$$

(Laplace's equation), which is the desired equation for the potential Φ . The sum of any solutions of (8) is a solution of (8) again, as can easily be seen. This is equivalent to the superposition of flows; the sum of the potential, of the impulsive pressures, or of the velocity components of several potential flows give a potential flow again.

All this refers originally to the case only that the flow is created by one impulsive pressure from rest. But every continuous and changing pressure can be replaced by infinitely many small impulsive pressures, and the resultant flow is the superposition of the flows created by each impulsive pressure. And as the superposition of potential flows gives a potential flow again, it is thus demonstrated that all aerodynamic flows are potential flows.

It can further be shown that for each motion of the bodies immersed in the fluid, there exists only one potential flow. For the integral (6) applied to a stream line (that is, a line always parallel to the velocity) has always the same sign of the integrant, and hence can not become zero. Hence a stream line can not be closed, as otherwise the integral (6) would give two different potentials for the same point, or different impulsive pressures, which is not possible. On the contrary, each stream line begins and ends at the surface of one of the immersed bodies. Now suppose that two potential flows exist for one motion of the bodies. Then reverse one of them by changing the sign of the potential and superpose it on the other. The resulting flow is characterized by all bodies being at rest. But then no stream line can begin at their surface, and hence the flow has no stream lines at all and the two original flows are demonstrated to be identical.

It remains to compute the pressure at each point of a potential flow. The acceleration of each particle is equal to the negative gradient of the pressure, divided by the density of the fluid. The pressure is therefore to be expressed as a function of the space coordinates, and so is the acceleration of a particle. Each component of the acceleration, say $\frac{du}{dt}$, has to be expressed by the local rate of change of the velocity component at a certain point $\frac{\partial u}{\partial t}$ and

by the velocity components and their local derivatives themselves. This is done by the equation

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \quad (9)$$

For during the unit of time the particle changes its coordinates by u , v , and w , respectively, and therefore reaches a region where the velocity is larger by $u \frac{\partial u}{\partial x}$, etc. This increase of velocity has to be added to the rate of change per unit time of the velocity at one particular point.

The general principles of mechanics, applied to a particle of unit volume, give therefore

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (10)$$

Substituting equation (7) in the last equation, we have

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + w \frac{\partial w}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (11)$$

Integrating this with respect to dx gives

$$\rho \frac{\partial \Phi}{\partial t} + \frac{\rho}{2} (u^2 + v^2 + w^2) = -\frac{1}{\rho} p \quad (12)$$

The equations for the two other components of the acceleration would give the same equation.

Hence it appears that the pressure can be divided into two parts superposed. The first part, $-\rho \frac{\partial \Phi}{\partial t}$, is the part of the pressure building up or changing the potential flow. It is zero if the flow is steady; that is, if

$$\frac{\partial \Phi}{\partial t} = 0 \quad (13)$$

The second part,

$$-\frac{\rho}{2} V^2 \quad (14)$$

if the pressure necessary to maintain and keep up the steady potential flow. It depends only on the velocity and density of the fluid. The greater the velocity, the smaller the pressure. It is sometimes called Bernoulli's pressure. This pressure acts permanently without changing the flow, and hence without changing its kinetic energy. It follows therefore that the Bernoulli's pressure (14) acting on the surface of a moving body, can not perform or consume any mechanical work. Hence in the case of the straight motion of a body the component of resultant force parallel to the motion is zero.

Some important formulas follow from the creation of the flow by the impulsive pressure $-\Phi\rho$. I will assume one body only, though this is not absolutely necessary for a part of the results. The distribution of this impulsive pressure over the surface of the bodies or body is characterized by a resultant impulsive force and a resultant impulsive moment. As further characteristic there is the mechanical work performed by the impulsive pressure during the creation of the flow, absorbed by the air and contained afterwards in the flow as kinetic energy of all particles.

It happens sometimes that the momentum imparted to the flow around a body moving translatory is parallel to the motion of the body. Since this momentum is proportional to the velocity, the effect of the air on the motion of the body in this direction is then taken care of by imparting to the body an apparent additional mass. If the velocity is not accelerated, no force is necessary to maintain the motion. The body experiences no drag, which is plausible, as no dissipation of energy is assumed. A similar thing may happen with a rotating body, where

then the body seems to possess an apparent additional moment of momentum. In general, however, the momentum imparted to the fluid is not parallel to the motion of the body, but it possesses a lateral component. The body in general possesses different apparent masses with respect to motions in different directions, and that makes the mechanics of a body surrounded by a perfect fluid different from that of one moving in a vacuum.

The kinetic energy imparted to the air is in a simple relation to the momentum and the velocity of the body. During the generation of the flow the body has the average velocity $\frac{V}{2}$ during the time dt , hence it moves through the distance $\frac{V}{2}dt$. The work performed is equal to the product of the component of resultant force of the creating pressure in the direction of motion, multiplied by this path, hence it is equal to half the product of the velocity and the component of the impulsive force in its direction.

The same argument can be used for the impulsive pressure acting over the surface of the body. Let dn be a linear element at right angles to the surface of the body drawn outward. The velocity at right angles to the surface is then, $-d\Phi/dn$ and the pressure $-\rho\Phi$ acts through the distance $-\frac{d\Phi/dn}{2}dt$. The work performed all over the surface is therefore

$$T = \int \frac{\rho}{2} \Phi \frac{d\Phi}{dn} dS \dots\dots\dots (15)$$

which integral is to be extended over the entire surface of the body consisting of all the elements dS . The expression under the integral contains the mass of the element of fluid displaced by the surface element of the body per unit of time, each element of mass multiplied by the velocity potential. The Bernoulli pressure does not perform any work, as discussed above, and is therefore omitted.

The apparent mass of a body moving in a particular direction depends on the density of the fluid. It is more convenient therefore to consider a volume of the fluid having a mass equal to the apparent mass of the body. This volume is

$$K = \frac{T}{V^2 \frac{\rho}{2}} \dots\dots\dots (16)$$

and depends only on the dimensions and form of the body.

The kinetic energy of the flow relative to a moving body in an infinite fluid is of course infinite. It is possible, however, to consider the diminution of the kinetic energy of the air moving with constant velocity brought about by the presence of a body at rest. This diminution of energy has two causes. The body displaces fluid, and hence the entire energy of the fluid is lessened by the kinetic energy of the displaced fluid. Further, the velocity of the air in the neighborhood of the body is diminished on the average. The forces between the body and the fluid are the same in both cases, whether the air or the body moves. Hence this second diminution of kinetic energy is equal to the kinetic energy of the flow produced by the moving body in the fluid otherwise at rest.

II. THE AERODYNAMIC FORCES ON AIRSHIP HULLS.

An important branch of theoretical aerodynamics deals with moments on bodies moving through the air while producing a potential flow. Wings produce a flow different from a potential flow, in the strict meaning of the word. The wings have therefore to be excluded from the following discussion.

Consider first bodies moving straight and with constant velocity V through air extending in all directions to infinity. There can not then be a drag, as the kinetic energy of the flow remains constant and no dissipation of energy is supposed to take place. Nor can there be a

lift in conformity with the remarks just made. Hence the air pressures can at best produce a resultant pure couple of forces or resultant moment. The magnitude and direction of this moment will depend on the magnitude of the velocity V and on the position of the body relative to the direction of its motion. With a change of velocity all pressures measured from a suitable standard, change proportional to the square of the velocity, as follows from equation (14). Hence the resultant moment is likewise proportional to the square of the velocity. In addition it will depend on the position of the body relative to the direction of motion. The study of this latter relation is the chief subject of this section. At each different position of the body relative to the motion the flow produced is different in general and so is the momentum of the flow, possessing different components in the direction of and at right angles to the direction of motion. By no means, however, can the relation between the momentum and the direction of motion be quite arbitrarily prescribed. The flow due to the straight motion in any direction can be obtained by the superposition of three flows produced by the motions in three particular directions. That restricts the possibilities considerably. But that is not all, the moments can not even arbitrarily be prescribed in three directions. I shall presently show that there are additional restrictions based on the principle of conservation of energy and momentum.

Let there be a component of the momentum lateral to the motion, equal to $K_3 V \rho$, where ρ denotes the density of the air. Since the body is advancing, this lateral component of the momentum has continually to be annihilated at its momentary position and to be created anew in its next position, occupied a moment later. This process requires a resultant moment

$$M = K_3 V^2 \rho \text{-----} (17)$$

about an axis at right angles to the direction of motion and to the momentum. In other words, the lateral component of the momentum multiplied by the velocity gives directly the resultant moment. Conversely, if the body experiences no resultant moment and hence is in equilibrium, the momentum of the air flow must be parallel to the motion.

Now consider a flow relative to the body with constant velocity V except for the disturbance of the body and let us examine its (diminution of) kinetic energy. If the body changes its position very slowly, so that the flow can still be considered as steady, the resultant moment is not affected by the rotation but is the same as corresponding to the momentary position and stationary flow. This moment then performs or absorbs work during the slow rotation. It either tends to accelerate the rotation, so that the body has to be braked, or it is necessary to exert a moment on the body in order to overcome the resultant moment. This work performed or absorbed makes up for the change of the kinetic energy of the flow. That gives a fundamental relation between the energy and the resultant moment.

There are as many different positions of the body relative to its motion as a sphere has radii. The kinetic energy of the flow is in general different for all directions, the velocity V and density ρ supposed to be constant. It has the same value, however, if the motion of the immersed solid is reversed, for then the entire flow is reversed. Therefore each pair of directions differing by 180° has the same kinetic energy. This energy moreover is always positive and finite. There must therefore be at least one pair of directions, where it is a minimum and one where it is a maximum. Moving parallel to either of these directions the body is in equilibrium and experiences no resultant moment. This follows from the consideration that then a small change in the direction of motion does not give rise to a corresponding change of the kinetic energy; the moment does not perform any work, and hence must be zero. The equilibrium is stable if the diminution of energy of the entire flow is a maximum and unstable if it is a minimum. It can be proved that in addition there must be at least one other axis of equilibrium. This is the position "neutral" with respect to the stable direction and at the same time neutral with respect to the unstable one. I call these directions "main axes."

I proceed to demonstrate that the three main axes of equilibrium are always at right angles to each other. Consider first the motion parallel to a plane through one of the main axes and

only the components of the momentum parallel to this plane. The direction of motion of the body may be indicated by the angle α in such a way that $\alpha=0$ is one motion of equilibrium, and hence without lateral component of momentum. The component of momentum in the direction of the motion may then (that is, when $\alpha=0$) be $K_1\rho V$. When moving at the angle of $\alpha=90^\circ$, the momentum may be supposed to possess the components $K_2\rho V$ parallel and $K_3\rho V$ at right angles to the motion, and we shall prove at once that the only momentum is the former.

The kinetic energy for any direction α can be written in the general form

$$T = V^2 \frac{\rho}{2} (K_1 \cos^2 \alpha + K_2 \sin^2 \alpha + K_3 \cos \alpha \sin \alpha)$$

and hence the resultant moment is

$$M = dT/d\alpha = V^2 \frac{\rho}{2} [(K_2 - K_1) \sin 2\alpha + K_3 \cos 2\alpha] \dots \dots \dots (18)$$

This resultant moment was supposed to be zero at $\alpha=0$. Hence $K_3=0$, and it follows that $\alpha=90^\circ$ is a position of equilibrium for motions in the plane considered. As for other motions, it is to be noticed that the third component of the momentum, at right angles to the plane, changes if the plane rotates around the axis of equilibrium. It necessarily changes its sign during a revolution, and while doing it M is zero. Thus it is demonstrated that there are at least two axes at right angles to each other where all lateral components of the momentum are zero, and hence the motion is in equilibrium. And as this argument holds true for any pair of the three axes of equilibrium, it is proved that there are always at least three axes of equilibrium at right angles to each other.

Resolving the velocity V of the body into three components, u, v, w , parallel to these three main axes, the kinetic energy can be expressed

$$\frac{\rho}{2} (K_1 u^2 + K_2 v^2 + K_3 w^2)$$

The differential of the energy

$$\rho (K_1 u du + K_2 v dv + K_3 w dw)$$

is identically zero in more than three pairs of positions only if at least two of the K 's are equal. Then it is zero in an infinite number of directions, and there are an infinite number of directions of equilibrium. The body is in equilibrium in all directions of motion only if all three K 's are equal; that is, if the apparent mass of the body is the same in all directions. That is a special case.

In all other cases the body experiences a resultant moment if moving with the velocity components u, v , and w parallel to the three main axes. The component of this resultant moment is determined by the momentary lateral momentum and its components, as stated in equation 17.

In most practical problems the motion occurs in a main plane; that is, at right angles to a main axis. Then the entire resultant moment is according to (17) the product of the velocity and the component of momentum at right angles to it, giving

$$M = V^2 \frac{\rho}{2} (K_2 - K_1) \sin 2\alpha \dots \dots \dots (19)$$

In general, the three main momenta of the flow, parallel to the respective motion, do not pass through one center. Practical problems occur chiefly with bodies of revolution. With them as well as with bodies with a center of symmetry—that is, such as have three planes of symmetry—the relation between the motion and the momenta is simple. It follows then from symmetry that the body possesses an aerodynamic center through which the three main momenta pass. This means that the body can be put into any straight motion by applying a force at a fixed

center. The force, however, is not parallel to the motion except in the main directions. The center where the force has to be applied coincides with the aerodynamic center, if the center of gravity of the body does so or if the mass of the body itself can be neglected compared with any of the three main additional masses.

Airship hulls are often bounded by surfaces of revolution. In addition they are usually rather elongated, and if the cross sections are not exactly round, they are at least approximately of equal and symmetrical shape and arranged along a straight axis. Surfaces of revolution have, of course, equal transverse apparent masses; each transverse axis at right angles to the axis of revolution is a main direction. For very elongated surfaces of revolution a further important statement may be made regarding the magnitude of the longitudinal and transverse apparent mass. When moving transversely the flow is approximately two-dimensional along the greatest part of the length. The apparent additional mass of a circular cylinder moving at right angles to its axis will be shown to be equal to the mass of the displaced fluid. It follows therefore that the apparent transverse additional mass of a very elongated body of revolution is approximately equal to the mass of the displaced fluid. It is slightly smaller, as near the ends the fluid has opportunity to pass the bow and stern. For cross sections other than circular the two main apparent masses follow in a similar way from the apparent mass of the cross section in the two-dimensional flow.

The longitudinal apparent additional mass, on the other hand, is small when compared with the mass of the displaced fluid. It can be neglected if the body is very elongated or can at least be rated as a small correction. This follows from the fact that only near the bow and the stern does the air have velocities of the same order of magnitude as the velocity of motion. Along the ship the velocity not only is much smaller but its direction is essentially opposite to the direction of motion, for the bow is continually displacing fluid and the stern makes room free for the reception of the same quantity of fluid. Hence the fluid is flowing from the bow to the stern, and as only a comparatively small volume is displaced per unit of time and the space is free in all directions to distribute the flow, the average velocity will be small.

It is possible to study this flow more closely and to prove analytically that the ratio of the apparent mass to the displaced mass approaches zero with increasing elongation. This proof, however, requires the study or knowledge of quite a number of conceptions and theorems, and it seems hardly worth while to have the student go through all this in order to prove such a plausible and trivial fact.

The actual magnitudes of the longitudinal and transverse masses of elongated surfaces of revolution can be studied by means of exact computations made by H. Lamb (reference 5), with ellipsoids of revolutions of different ratio of elongation. The figures of k_1 and k_2 , where $K = k \times \text{volume}$, obtained by him are contained in Table I of this paper, and $k_1 - k_2$ is computed. For bodies of a shape reasonably similar to ellipsoids it can be approximately assumed that $(k_1 - k_2)$ has the same value as for an ellipsoid of the same length and volume; that is, if Vol/L^3 has the same value.

The next problem of interest is the resultant aerodynamic force if the body rotates with constant velocity around an axis outside of itself. That is now comparatively simple, as the results of the last section can be used. The configuration of flow follows the body, with constant shape, magnitude, and hence with constant kinetic energy. The resultant aerodynamic force, therefore, must be such as neither to consume nor to perform mechanical work. This leads to the conclusion that the resultant force must pass through the axis of rotation. In general it has both a component at right angles and one parallel to the motion of the center of the body.

I confine the investigation to a surface of revolution. Let an airship with the apparent masses $K_{1\rho}$ and $K_{2\rho}$ and the apparent moment of inertia $K'\rho$ for rotation about a transverse axis through its aerodynamic center move with the velocity V of its aerodynamic center around an axis at the distance r from its aerodynamic center and let the angle of yaw ϕ be measured between the axis of the ship and the tangent of the circular path at the aerodynamic center. The ship is then rotating with the constant angular velocity V/r . The entire motion can be obtained by superposition of the longitudinal motion $V \cos \phi$ of the aerodynamic center, the

transverse velocity $V \sin \phi$, and the angular velocity V/r . The longitudinal component of the momentum is $V\rho \cos \phi \cdot k_1 \cdot \text{vol}$, and the transverse component of the momentum is $V\rho \sin \phi \cdot k_2 \cdot \text{vol}$. Besides, there is a moment of momentum due to the rotation. This can be expressed by introducing the apparent moment of inertia $K'\rho = k'J\rho$ where J is the moment of inertia of the displaced air; thus making the angular momentum

$$k'J\rho \left(\frac{V}{r} \right)$$

As it does not change, it does not give rise to any resultant aerodynamic force or moment during the motion under consideration.

The momentum remains constant, too, but changes its direction with the angular velocity V/r . This requires a force passing through the center of turn and having the transverse component

$$F_t = K_1\rho \cos \phi V^2/r \dots\dots\dots (20)$$

and the longitudinal component

$$F_l = K_2\rho \sin \phi V^2/r \dots\dots\dots (21)$$

The first term is almost some kind of centrifugal force. Some air accompanies the ship, increasing its longitudinal mass and hence its centrifugal force. It will be noticed that with actual airships this additional centrifugal force is small, as k_1 is small. The force attacking at the center of the turn can be replaced by the same force attacking at the aerodynamic center and a moment around this center of the magnitude.

$$M = \frac{1}{2}(K_2 - K_1)\rho \sin 2\phi V^2 \dots\dots\dots (22)$$

This moment is equal in direction and magnitude to the unstable moment found during straight motion under the same angle of pitch or yaw. The longitudinal force is in practice a negative drag as the bow of the ship is turned toward the inside of the circle. It is of no great practical importance as it does not produce considerable structural stresses.

It appears thus that the ship when flying in a curve or circle experiences almost the same resultant moment as when flying straight and under the same angle of pitch or yaw. I proceed to show, however, that the transverse aerodynamic forces producing this resultant moment are distributed differently along the axis of the ship in the two cases.

The distribution of the transverse aerodynamic forces along the axis can conveniently be computed for very elongated airships. It may be supposed that the cross section is circular, although it is easy to generalize the proceeding for a more general shape of the cross section.

The following investigation requires the knowledge of the apparent additional mass of a circular cylinder moving in a two-dimensional flow. I proceed to show that this apparent additional mass is exactly equal to the mass of the fluid displaced by the cylinder. In the two-dimensional flow the cylinder is represented by a circle.

Let the center of this circle coincide with the origin of a system of polar coordinates R and ϕ , moving with it, and let the radius of the circle be denoted by r . Then the velocity potential of the flow created by this circle moving in the direction $\phi = 0$ with the velocity v is $\Phi = vr^2 (\cos \phi)/R$. For this potential gives the radial velocity components

$$\frac{d\Phi}{dR} = -v \frac{r^2}{R^2} \cos \phi$$

and at the circumference of the circle this velocity becomes $v \cos \phi$. This is in fact the normal component of velocity of a circle moving with the velocity v in the specified direction.

The kinetic energy of this flow is now to be determined. In analogy to equation (15), this is done by integrating along the circumference of the circle the product of (a) the elements of half the mass of the fluid penetrating the circle $\left(\frac{\rho}{2} \cos \phi r d\phi \right)$ and (b), the value of the veloc-

ity potential at that point $(-v \cos \phi \cdot r)$. The integral is therefore

$$\frac{\rho}{2} \int_0^{2\pi} \cos^2 \phi v^2 r^2 d\phi$$

giving the kinetic energy $r^2 \pi v^2 \frac{\rho}{2}$.

This shows that in fact the area of apparent mass is equal to the area of the circle.

I am now enabled to return to the airship.

If a very elongated airship is in translatory horizontal motion through air otherwise at rest and is slightly pitched, the component of the motion of the air in the direction of the axis of the ship can be neglected. The air gives way to the passing ship by flowing around the axis of the ship, not by flowing along it. The air located in a vertical plane at right angles to the motion remains in that plane, so that the motion in each plane can be considered to be two-dimensional. Consider one such approximately vertical layer of air at right angles to the axis while the ship is passing horizontally through it. The ship displaces a circular portion of this layer, and this portion changes its position and its size. The rate of change of position is expressed by an apparent velocity of this circular portion, the motion of the air in the vertical layer is described by the two-dimensional flow produced by a circle moving with the same velocity. The momentum of this flow is $Svpdx$, where S is the area of the circle, and v the vertical velocity of the circle, and dx the thickness of the layer. Consider first the straight flight of the ship under the angle of pitch ϕ . The velocity v of the displaced circular portion of the layer is then constant over the whole length of the ship and is $V \sin \phi$, where V is the velocity of the airship along the circle. Not so the area S ; it changes along the ship. At a particular layer it changes with the rate of change per unit time,

$$V \cos \phi \cdot \frac{dS}{dx}$$

where x denotes the longitudinal coordinate.

Therefore the momentum changes with the rate of change

$$V^2 \frac{\rho}{2} \sin 2\phi \frac{dS}{dx} dx$$

This gives a down force on the ship with the magnitude

$$dF = dx V^2 \frac{\rho}{2} \sin 2\phi \frac{dS}{dx} \dots \dots \dots (23)$$

Next, consider the ship when turning, the angle of yaw being ϕ . The momentum in each layer is again

$$vSpdx$$

The transverse velocity v is now variable, too, as it is composed of the constant portion $V \sin \phi$, produced by the yaw, and of the variable portion $V \frac{x}{r} \cos \phi$, produced by the turning. $x=O$ represents the aerodynamic center. Hence the rate of change of the momentum per unit length is

$$V^2 \frac{\rho}{2} \sin 2\phi \frac{dS}{dx} + \rho \frac{V^2}{r} \cos \phi \frac{d}{dx} (xS)$$

giving rise to the transverse force per unit length

$$V^2 \frac{\rho}{2} \sin 2\phi \frac{dS}{dx} + V^2 \frac{\rho}{r} \cos \phi \left(S + x \frac{dS}{dx} \right)$$

or otherwise written

$$dF = dx \left(V^2 \frac{\rho}{2} \sin 2\phi \frac{dS}{dx} + V^2 \frac{\rho}{r} \cos \phi S + V^2 \frac{\rho}{r} \cos \phi x \frac{dS}{dx} \right) \dots \dots \dots (24)$$

The first term agrees with the moment of the ship flying straight having a pitch ϕ . The direction of this transverse force is opposite at the two ends, and gives rise to an unstable moment. The ships in practice have the bow turned inward when they fly in turn. Then the transverse force represented by the first term of (24) is directed inward near the bow and outward near the stern.

The sum of the second and third terms of (24) gives no resultant force or moment. The second term alone gives a transverse force, being in magnitude and distribution almost equal to the transverse component of the centrifugal force of the displaced air, but reversed. This latter becomes clear at the cylindrical portion of the ship, where the two other terms are zero. The front part of the cylindrical portion moves toward the center of the turn and the rear part moves away from it. The inward momentum of the flow has to change into an outward momentum, requiring an outward force acting on the air, and giving rise to an inward force reacting this change of momentum.

The third term of (24) represents forces almost concentrated near the two ends and their sum in magnitude and direction is equal to the transverse component of the centrifugal force of the displaced air. They are directed outward.

Ships only moderately elongated have resultant forces and a distribution of them differing from those given by the formulas (23) and (24). The assumption of the layers remaining plane is more accurate near the middle of the ship than near the ends, and in consequence the transverse forces are diminished to a greater extent at the ends than near the cylindrical part when compared with the very elongated hulls. In practice, however, it will often be exact enough to assume the same shape of distribution for each term and to modify the transverse forces by constant diminishing factors. These factors are logically to be chosen different for the different terms of (24). For the first term represents the forces giving the resultant moment proportional to $(k_2 - k_1)$, and hence it is reasonable to diminish this term by multiplying it by $(k_2 - k_1)$. The second and third terms take care of the momenta of the air flowing transverse with a velocity proportional to the distance from the aerodynamic center. The moment of inertia of the momenta really comes in, and therefore it seems reasonable to diminish these terms by the factor k' , the ratio of the apparent moment of inertia to the moment of inertia of the displaced air.

The transverse component of the centrifugal force produced by the air taken along with the ship due to its longitudinal mass is neglected. Its magnitude is small; the distribution is discussed in reference (3) and may be omitted in this treatise.

The entire transverse force on an airship, turning under an angle of yaw with the velocity V and a radius r , is, according to the preceding discussion,

$$dF = dx \left[(k_2 - k_1) \frac{dS}{dx} V^2 \frac{\rho}{2} \sin 2\phi + k' V \frac{\rho}{r} S \cos \phi + k' V^2 \frac{\rho \cdot x}{r} \frac{dS}{dx} \cos \phi \right] \dots \dots \dots (25)$$

This expression does not contain of course the air forces on the fins.

In the first two parts of this paper I discussed the dynamical forces of bodies moving along a straight or curved path in a perfect fluid. In particular I considered the case of a very elongated body and as a special case again one bounded by a surface of revolution.

The hulls of modern rigid airships are mostly surfaces of revolution and rather elongated ones, too. The ratio of the length to the greatest diameter varies from 6 to 10. With this elongation, particularly if greater than 8, the relations valid for infinite elongation require only a small correction, only a few per cent, which can be estimated from the case of ellipsoids for which the forces are known for any elongation. It is true that the transverse forces are not only increased or decreased uniformly, but also the character of their distribution is slightly changed. But this can be neglected for most practical applications, and especially so since there are other differences between theoretical and actual phenomena.

Serious differences are implied by the assumption that the air is a perfect fluid. It is not, and as a consequence the air forces do not agree with those in a perfect fluid. The resulting air force by no means gives rise to a resulting moment only; it is well known that an airship

hull model without fins experiences both a drag and a lift, if inclined. The discussion of the drag is beyond the scope of this paper. The lift is very small, less than 1 per cent of the lift of a wing with the same surface area. But the resulting moment is comparatively small, too, and therefore it happens that the resulting moment about the center of volume is only about 70 per cent of that expected in a perfect fluid. It appears, however, that the actual resulting moment is at least of the same range of magnitude, and the contemplation of the perfect fluid gives therefore an explanation of the phenomenon. The difference can be explained. The flow is not perfectly irrotational, for there are free vortices near the hull, especially at its rear end, where the air leaves the hull. They give a lift acting at the rear end of the hull, and hence decreasing the unstable moment with respect to the center of volume

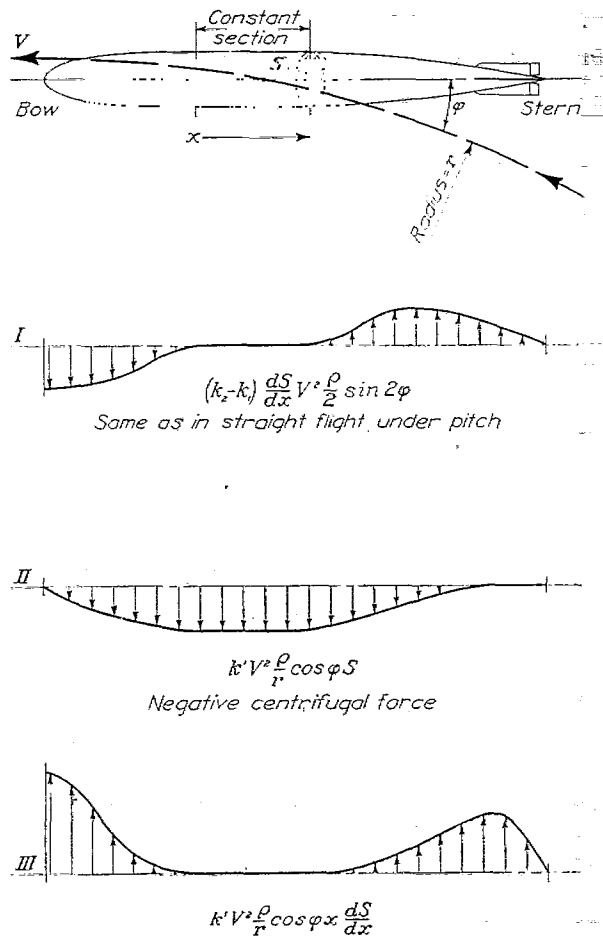


FIG. 1.—Diagram showing the direction of the transverse air forces acting on an airship flying in a turn. The three terms are to be added together.

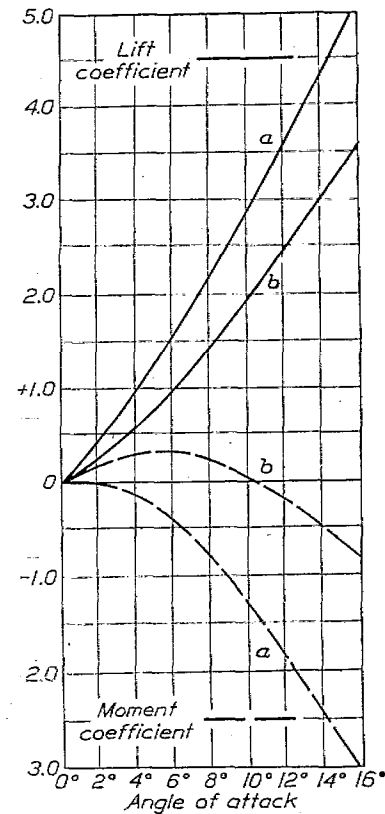


FIG. 2.

What is perhaps more important, they produce a kind of induced downwash, diminishing the effective angle of attack, and hence the unstable moment.

This refers to airship hulls without fins, which are of no practical interest. Airship hulls with fins must be considered in a different way. The fins are a kind of wings; and the flow around them, if they are inclined, is far from being even approximately irrotational and their lift is not zero. The circulation of the inclined fins is not zero; and as they are arranged in the rear of the ship, the vertical flow induced by the fins in front of them around the hull is directed upward if the ship is nosed up. Therefore the effective angle of attack is increased, and the influence of the lift of the hull itself is counteracted. For this reason it is to be expected that the transverse forces of hulls with fins in air agree better with these in a perfect fluid. Some model tests to be discussed now confirm this.

These tests give the lift and the moment with respect to the center of volume at different angles of attack and with two different sizes of fins. If one computes the difference between the observed moment and the expected moment of the hull alone, and divides the difference by the observed lift, the apparent center of pressure of the lift of the fins results. If the center of pressure is situated near the middle of the fins, and it is, it can be inferred that the actual flow of the air around the hull is not very different from the flow of a perfect fluid. It follows, then, that the distribution of the transverse forces in a perfect fluid gives a good approximation of the actual distribution, and not only for the case of straight flight under consideration, but also if the ship moves along a circular path.

The model tests which I proceed to use were made by Georg Fuhrmann in the old Goettingen wind tunnel and published in the *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, 1910. The model, represented in Figure 3, had a length of 1,145 millimeters, a maximum diameter of 188 millimeters, and a volume of 0.0182 cubic meter. Two sets of fins were attached to the hull, one after another; the smaller fins were rectangular, 6.5 by 13 centimeters, and the larger ones, 8 by 15 centimeters. $(\text{Volume})^{2/3} = 0.069$ square meter. In Figure 3 both fins are shown. The diagram in Figure 2 gives both the observed lift and the moment expressed by means of absolute coefficients. They are reduced to the unit of the dynamical pressure, and also the moment is reduced to the unit of the volume, and the lift to the unit of $(\text{volume})^{2/3}$.

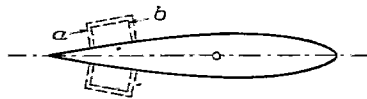


FIG. 3.—Airship model.

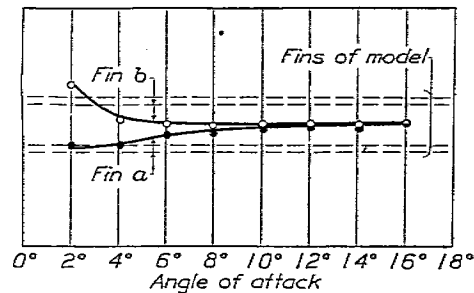


FIG. 4.—Center of pressure of fin forces.

Diagram Figure 4 shows the position of the center of pressure computed as described before. The two horizontal lines represent the leading and the trailing end of the fins. It appears that for both sizes of the fins the curves nearly agree, particularly for greater angles of attack at which the tests are more accurate. The center of pressure is situated at about 40 per cent of the chord of the fins. I conclude from this that the theory of a perfect fluid gives a good indication of the actual distribution of the transverse forces. In view of the small scale of the model, the agreement may be even better with actual airships.

III. SOME PRACTICAL CONCLUSIONS.

The last examination seems to indicate that the actual unstable moment of the hull in air agrees nearly with that in a perfect fluid. Now the actual airships with fins are statically unstable (as the word is generally understood, not aerostatically of course), but not much so, and for the present general discussion it can be assumed that the unstable moment of the hull is nearly neutralized by the transverse force of the fins. I have shown that this unstable moment is $M = (\text{volume}) (k_2 - k_1) V^2 \frac{\rho}{2} \sin 2\phi$, where $(k_2 - k_1)$ denotes the factor of correction due to finite elongation. Its magnitude is discussed in the first part of this paper. Hence the transverse force of the fins must be about $\frac{M}{a}$, where a denotes the distance between the fin and the center of gravity of the ship. Then the effective area of the fins—that is, the area of a wing giving the same lift in a two-dimensional flow—follows:

$$\frac{(\text{Volume})(k_2 - k_1)}{a\pi}$$

Taking into account the span b of the fins—that is, the distance of two utmost points of a pair of fins—the effective fin area S must be

$$\frac{(\text{Volume})(k_2 - k_1)}{a} \times \frac{1 + 2 \frac{S}{b^2}}{\pi}$$

This area S , however, is greater than the actual fin area. Its exact size is uncertain, but a far better approximation than the fin area is obtained by taking the projection of the fins and the part of the hull between them. This is particularly true if the diameter of the hull between the fins is small.

If the ends of two airships are similar, it follows that the fin area must be proportional to $(k_2 - k_1)(\text{volume})/a$. For rather elongated airships $(k_2 - k_1)$ is almost equal to 1 and constant, and for such ships therefore it follows that the fin area must be proportional to $(\text{volume})/a$, or, less exactly, to the greatest cross section, rather than to $(\text{volume})^{2/3}$. Comparatively short ships, however, have a factor $(k_2 - k_1)$ rather variable, and with them the fin area is more nearly proportional to $(\text{volume})^{2/3}$.

This refers to circular section airships. Hulls with elliptical section require greater fins parallel to the greater plan view. If the greater axis of the ellipse is horizontal, such ships are subjected to the same bending moments for equal lift and size, but the section modulus is smaller, and hence the stresses are increased. They require, however, a smaller angle of attack for the same lift. The reverse holds true for elliptical sections with the greater axes vertical.

If the airship flies along a circular path, the centrifugal force must be neutralized by the transverse force of the fin, for only the fin gives a considerable resultant transverse force. At the same time the fin is supposed nearly to neutralize the unstable moment. I have shown now that the angular velocity, though indeed producing a considerable change of the distribution of the transverse forces, and hence of the bending moments, does not give rise to a resulting force or moment. Hence, the ship flying along the circular path must be inclined by the same angle of yaw as if the transverse force is produced during a rectilinear flight by pitching. From the equation of the transverse force

$$\text{Vol } \rho \frac{V^2}{r} = \frac{\text{Vol}(k_2 - k_1) \frac{V^2 \rho}{2} \sin 2\phi}{a}$$

it follows that the angle is approximately

$$\phi \sim \frac{a}{r} \frac{1}{k_2 - k_1}$$

This expression in turn can be used for the determination of the distribution of the transverse forces due to the inclination. The resultant transverse force is produced by the inclination of the fins. The rotation of the rudder has chiefly the purpose of neutralizing the damping moment of the fins themselves.

From the last relation, substituted in equation (25), follows approximately the distribution of the transverse forces due to the inclination of pitch, consisting of

$$\frac{dS}{dx} V^2 \rho \frac{2a}{2r} dx \dots \dots \dots (26)$$

This is only one part of the transverse forces. The other part is due to the angular velocity; it is approximately

$$k' \frac{2x}{r} \frac{dS}{dx} V^2 \rho \frac{2a}{2} dx + k' \frac{V^2 \rho}{r} S dx \dots \dots \dots (27)$$

The first term in (27) together with (26) gives a part of the bending moment. The second term in (27), having mainly a direction opposite to the first one and to the centrifugal force, is almost neutralized by the centrifugal forces of the ship and gives additional bending moments not very considerable either. It appears, then, that the ship experiences smaller bending moments when creating an air force by yaw opposite to the centrifugal force than when creating the same

transverse force during a straight flight by pitch. For ships with elliptical sections this can not be said so generally. The second term in (27) will then less perfectly neutralize the centrifugal force, if that can be said at all, and the bending moments become greater in most cases.

Most airship pilots are of the opinion that severe aerodynamic forces act on airships flying in bumpy weather. An exact computation of the magnitude of these forces is not possible, as they depend on the strength and shape of the gusts and as probably no two exactly equal gusts occur. Nevertheless, it is worth while to reflect on this phenomenon and to get acquainted with the underlying general mechanical principles. It will be possible to determine how the magnitude of the velocity of flight influences the air forces due to gusts. It even becomes possible to estimate the magnitude of the air forces to be expected, though this estimation will necessarily be somewhat vague, due to ignorance of the gusts.

The airship is supposed to fly not through still air but through an atmosphere the different portions of which have velocities relative to each other. This is the cause of the air forces in bumpy weather, the airship coming in contact with portions of air having different velocities. Hence, the configuration of the air flow around each portion of the airship is changing as it always has to conform to the changing relative velocity between the portion of the airship and the surrounding air. A change of the air forces produced is the consequence.

Even an airship at rest experiences aerodynamical forces in bumpy weather, as the air moves toward it. This is very pronounced near the ground, where the shape of the surrounding objects gives rise to violent local motions of the air. The pilots have the impression that at greater altitudes an airship at rest does not experience noticeable air forces in bumpy weather. This is plausible. The hull is struck by portions of air with relatively small velocity, and as the forces vary as the square of the velocity they can not become large.

It will readily be seen that the moving airship can not experience considerable air forces if the disturbing air velocity is in the direction of flight. Only a comparatively small portion of the air can move with a horizontal velocity relative to the surrounding air and this velocity can only be small. The effect can only be an air force parallel to the axis of the ship which is not likely to create large structural stresses.

There remains, then, as the main problem the airship in motion coming in contact with air moving in a transverse direction relative to the air surrounding it a moment before. The stresses produced are severer if a larger portion of air moves with that relative velocity. It is therefore logical to consider portions of air large compared with the diameter of the airship; smaller gusts produce smaller air forces. It is now essential to realize that their effect is exactly the same as if the angle of attack of a portion of the airship is changed. The air force acting on each portion of the airship depends on the relative velocity between this portion and the surrounding air. A relative transverse velocity u means an effective angle of attack of that portion equal to u/V , where V denotes the velocity of flight. The airship therefore is now to be considered as having a variable effective angle of attack along its axis. The magnitude of the superposed angle of attack is u/V , where u generally is variable. The air force produced at each portion of the airship is the same as the air force at that portion if the entire airship would have that particular angle of attack.

The magnitude of the air force depends on the conicity of the airship portion as described in section 2. The force is proportional to the angle of attack and to the square of the velocity of flight. In this case, however, the superposed part of the angle of attack varies inversely as the velocity of flight. It results, then, that the air forces created by gusts are directly proportional to the velocity of flight. Indeed, as I have shown, they are proportional to the product of the velocity of flight and the transverse velocity relative to the surrounding air.

A special and simple case to consider for a closer investigation is the problem of an airship immersing from air at rest into air with constant transverse horizontal or vertical velocity. The portion of the ship already immersed has an angle of attack increased by the constant amount u/V . Either it can be assumed that by operation of the controls the airship keeps its course or, better, the motion of an airship with fixed controls and the air forces acting on it under these conditions can be investigated. As the fins come under the influence of the increased

transverse velocity later than the other parts, the airship is, as it were, unstable during the time of immersing into the air of greater transverse velocity and the motion of the airship aggravates the stresses.

In spite of this the actual stresses will be of the same range of magnitude as if the airship flies under an angle of pitch of the magnitude u/V , for in general the change from smaller to greater transverse velocity will not be so sudden and complete as supposed in the last paragraph. It is necessary chiefly to investigate the case of a vertical transverse relative velocity u , for the severest condition for the airship is a considerable angle of pitch, and a vertical velocity u increases these stresses. Hence it would be extremely important to know the maximum value of this vertical velocity. The velocity in question is not the greatest vertical velocity of portions of the atmosphere occurring, but differences of this velocity within distances smaller than the length of the airship. It is very difficult to make a positive statement as to this velocity, but it is necessary to conceive an idea of its magnitude, subject to a correction after the question is studied more closely. Studying the meteorological papers in the reports of the British Advisory Committee for Aeronautics, chiefly those of 1909-10 and 1912-13, I should venture to consider a sudden change of the vertical velocity by 2 m./sec. (6.5 ft./sec.) as coming near to what to expect in very bumpy weather. The maximum dynamic lift of an airship is produced at low velocity, and is the same as if produced at high velocity at a comparatively low angle of attack, not more than 5° . If the highest velocity is 30 m./sec. (67 mi./hr.), the angle of attack u/V , repeatedly mentioned before, would be $\frac{57.3 \times 2}{30} = 3.8^\circ$. This is a little smaller than 5° , but the assumption for u is rather vague. It can only be said that the stresses due to gusts are of the same range of magnitude as the stresses due to pitch, but they are probably not larger.

A method for keeping the stresses down in bumpy weather is by slowing down the speed of the airship. This is a practice common among experienced airship pilots. This procedure is particularly recommended if the airship is developing large dynamic lift, positive or negative, as then the stresses are already large.

Length (diameters).	K_1 (longi- tudinal).	K_2 (trans- verse).	$K_2 - K_1$.	K' (rota- tion).
1	0.500	0.500	0	0
1.50	.305	.621	.316	.094
2.00	.209	.702	.493	.240
2.51	.156	.763	.607	.367
2.99	.122	.803	.681	.465
3.99	.082	.860	.778	.608
4.99	.059	.895	.836	.701
6.01	.045	.918	.873	.764
6.97	.036	.933	.897	.805
8.01	.029	.945	.916	.840
9.02	.024	.954	.930	.865
9.97	.021	.960	.939	.883
∞	.000	1.000	1.000	1.000

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