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MECHANICAL EFFICIENCY ANALYSIS OF A CARDAN JOINT WITH MANUFACTURING TOLERANCES

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Introduction

Cardan joints are common devices for transmitting the motion between misaligned intersecting axes. Their capability of easy mounting, of resisting high loads and commercial availability makes them an attractive solution, as a substitute of spherical pairs, in parallel robots.

Although their structure has been known for centuries, only recently a complete dynamic analysis has been presented in a series of papers authored by F. Freudenstein and his coworkers [1, 2, 3, 4]. In the mentioned references friction is not included.

Likely the first scientific contribution on the mechanical efficiency of Cardan joints is due to Morecki [5]. This model, based on a simplified static analysis and including the losses in the yoke bearings only, has been verified with a different analytical approach and refined by including also the losses in the fixed bearings. The results have been plotted in a design chart which allow to compute the efficiency as a function of the angle between input-output shaft axes [8].

The modeling of manufacturing errors in Cardan joints is introduced by considering a kinematically equivalent RCCC mechanism. In this paper a kinematic and static analysis of the RCCC mechanism by means of the dual numbers algebra is carried out first. Then, the effects of friction are included. For this purpose, the following hypotheses are adopted:

- Coulomb friction;
- absence of stiction;
- negligible inertia forces;
- absence of backlash in the kinematic pairs;
- rigid bodies.

As a main contribution, it is herein presented a model for computing the mechanical efficiency of this mechanism. To the best of the author's knowledge, in the scientific literature is not addressed the problem of computing the mechanical efficiency of Cardan joints in presence of manufacturing errors.

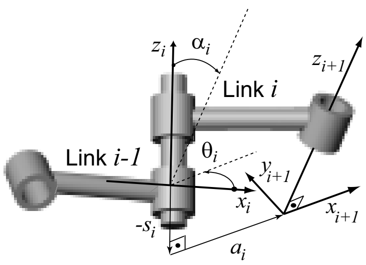

Figure 1: Denavit-Hartenberg parameters

Nomenclature

- a_i : minimum distance between z_i and z_{i+1} axes;
- D: distance between the bearings of the cross;
- d_i: diameter of shaft i;
- f: friction coefficient;
- F_{ix},F_{iy},F_{iz} joint forces cartesian components at the ith joint;

- F_{ix},F_{iy},F_{iz} joint forces at the ith joint;
- $x_iy_iz_i$: moving cartesian system attached to the i^{th} body, as in the Denavit-Hartenberg convention;
- s_i : relative linear displacement of the links measured according to the Denavit-Hartenberg convention (see Figure 1);
- α_i angle between z_i and z_{i+1} axes;
- ε dual unity $\left(\varepsilon^2 = 0\right)$
- η mechanical efficiency of the Cardan jojnt;
- ω_i: angular velocity of the i^{th} body, measured in the cartesian system $o - x_iy_iz_i$;
- θ_i : relative angular displacement of the links measured according to the Denavit-Hartenberg convention (see Figure 1);
- $\tau_f^{(i)}$: frictional torque at the i^{th} revolute joint;
- the $\widehat{}$ denote dual quantities;
- Dots denote differentiation w.r.t. time.

Kinematic analysis of the RCCC linkage

Let us denote with

$$\widehat{\theta}_i = \theta_i + \varepsilon s_i \ , \tag{1}$$
$$\widehat{\alpha}_i = \alpha_i + \varepsilon a_i \ , \tag{2}$$

the dual numbers which define, respectively, the relative position between adjacent links and the geometry of the ith link.

With reference to Figure 1, the transform matrix from coordinate system $o_{i+1} - x_{i+1}y_{i+1}z_{i+1}$ to $o_i - x_iy_iz_i$, in terms of such numbers is given by[1]

$$\left[\widehat{A}\right]_{i+1}^{i} = \begin{bmatrix} c\widehat{\theta}_i & -c\widehat{\alpha}_i s\widehat{\theta}_i & s\widehat{\alpha}_i s\widehat{\theta}_i \\ s\widehat{\theta}_i & c\widehat{\alpha}_i c\widehat{\theta}_i & -s\widehat{\alpha}_i c\widehat{\theta}_i \\ 0 & s\widehat{\alpha}_i & c\widehat{\alpha}_i \end{bmatrix} . \tag{3}$$

The closure condition for the RCCC mechanism shown in Figure 2 is expressed by the matrix product

$$\left[\widehat{A}\right]_2^1 \left[\widehat{A}\right]_3^2 \left[\widehat{A}\right]_4^3 \left[\widehat{A}\right]_4^1 = [I] \ , \tag{4}$$

where $[I]$ is the identity matrix.

The above equation can be rewritten in the form

$$\left[\widehat{A}\right]_3^2 \left[\widehat{A}\right]_4^3 = \left[\widehat{A}^T\right]_2^1 \left[\widehat{A}^T\right]_4^1 . \tag{5}$$

Carrying out the matrix products and equating the elements on the same rows and columns one obtains [4]:

$$\widehat{D}\sin\widehat{\theta}_4 + \widehat{E}\cos\widehat{\theta}_4 + \widehat{F} = 0 \ , \tag{6}$$

where

$$\widehat{D} = s\widehat{\alpha}_1 s\widehat{\alpha}_3 s\widehat{\theta}_1 \ , \tag{7}$$
$$\widehat{E} = -s\widehat{\alpha}_3 \left(c\widehat{\alpha}_1 s\widehat{\alpha}_4 + s\widehat{\alpha}_1 c\widehat{\alpha}_4 c\widehat{\theta}_1\right) \ , \tag{8}$$
$$\widehat{F} = -c\widehat{\alpha}_2 + c\widehat{\alpha}_3 \left(c\widehat{\alpha}_1 c\widehat{\alpha}_4 - s\widehat{\alpha}_1 s\widehat{\alpha}_4 c\widehat{\theta}_1\right) \ , \tag{9}$$

and

$$s\widehat{\theta}_2 = \frac{s\widehat{\theta}_1\left(c\widehat{\alpha}_3 s\widehat{\alpha}_4 + s\widehat{\alpha}_3 c\widehat{\alpha}_4 c\widehat{\theta}_4\right) + s\widehat{\alpha}_3 c\widehat{\theta}_1 s\widehat{\theta}_4}{s\widehat{\alpha}_2} \ , \tag{10}$$
$$c\widehat{\theta}_2 = \frac{c\widehat{\alpha}_1 c\widehat{\alpha}_2 - c\widehat{\alpha}_3 c\widehat{\alpha}_4 + s\widehat{\alpha}_3 s\widehat{\alpha}_4 c\widehat{\theta}_4}{s\widehat{\alpha}_1 s\widehat{\alpha}_2} \ , \tag{11}$$
$$s\widehat{\theta}_3 = \frac{s\widehat{\alpha}_1\left(s\widehat{\theta}_1 c\widehat{\theta}_4 + c\widehat{\alpha}_4 c\widehat{\theta}_1 s\widehat{\theta}_4\right) + c\widehat{\alpha}_1 s\widehat{\alpha}_4 s\widehat{\theta}_4}{s\widehat{\alpha}_2} \ , \tag{12}$$
$$c\widehat{\theta}_3 = \frac{s\widehat{\alpha}_1 s\widehat{\alpha}_4 c\widehat{\theta}_1 + c\widehat{\alpha}_2 c\widehat{\alpha}_3 - c\widehat{\alpha}_1 c\widehat{\alpha}_4}{s\widehat{\alpha}_2 s\widehat{\alpha}_3} \ . \tag{13}$$

Thus, the dual angles $\widehat{\theta}_2$, $\widehat{\theta}_3$ and $\widehat{\theta}_4$ are computed as follows [2]:

$$\widehat{\theta}_4 = 2\tan^{-1}\frac{-\widehat{D} \pm \sqrt{\widehat{D}^2 + \widehat{E}^2 - \widehat{F}^2}}{\widehat{F} - \widehat{E}} \ , \tag{14}$$
$$\widehat{\theta}_2 = \text{ATAN2}\left(\sin\widehat{\theta}_2, \cos\widehat{\theta}_2\right) \ , \tag{15}$$
$$\widehat{\theta}_3 = \text{ATAN2}\left(\sin\widehat{\theta}_3, \cos\widehat{\theta}_3\right) \ . \tag{16}$$

[1] c = cos and s = sin

[2] ATAN2 functions with dual numbers as arguments can be computed by means of the procedure presented in the Appendix.

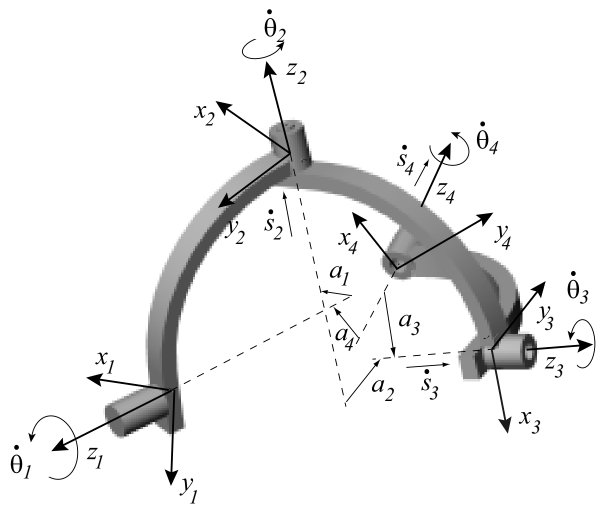

Figure 2: The RCCC kinematically equivalent linkage

Static analysis of the RCCC linkage

Let us denote with the dual numbers

$$\hat{F}_{xj} = F_{xj} + \varepsilon M_{xj}\ , \tag{17}$$

$$\hat{F}_{yj} = F_{yj} + \varepsilon M_{yj}\ , \tag{18}$$

$$\hat{F}_{zj} = F_{zj} + \varepsilon M_{zj}\ , \tag{19}$$

for $j = 1,2,3,4$, the joint forces.

Imposing the static equilibrium of the links one obtains[3]:

$$\begin{aligned}\hat{F}_{x1} = {} & \frac{c\hat{\theta}_1 s\hat{\alpha}_2 c\hat{\theta}_2 - s\hat{\alpha}_2 s\hat{\theta}_2 c\hat{\alpha}_1 s\hat{\theta}_1}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z1} \\ & + \frac{s\hat{\alpha}_2 s\hat{\theta}_1 s\hat{\theta}_2 - c\hat{\theta}_1 c\hat{\theta}_2 c\hat{\alpha}_1 s\hat{\alpha}_2 - c\hat{\theta}_1 c\hat{\alpha}_2 s\hat{\alpha}_1}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} \\ & + \frac{c\hat{\theta}_1}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{20}$$

$$\begin{aligned}\hat{F}_{y1} = {} & \frac{c\hat{\theta}_1 s\hat{\theta}_2 c\hat{\alpha}_1 s\hat{\alpha}_2 + s\hat{\theta}_1 s\hat{\alpha}_2 c\hat{\theta}_2}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z1} \\ & - \frac{s\hat{\theta}_1 s\hat{\alpha}_2 c\hat{\theta}_2 c\hat{\alpha}_1 + s\hat{\alpha}_1 s\hat{\theta}_1 c\hat{\alpha}_2 + s\hat{\alpha}_2 s\hat{\theta}_2 c\hat{\theta}_1}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} \\ & + \frac{s\hat{\theta}_1}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{21}$$

$$\begin{aligned}\hat{F}_{x2} = {} & \frac{s\hat{\alpha}_2 c\hat{\theta}_2}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z1} \\ & - \frac{s\hat{\alpha}_1 c\hat{\alpha}_2 + s\hat{\alpha}_2 c\hat{\theta}_2 c\hat{\alpha}_1}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} + \frac{\hat{F}_{z3}}{s\hat{\alpha}_2 s\hat{\theta}_2}\end{aligned} \tag{22}$$

$$\hat{F}_{y2} = \frac{\hat{F}_{k1}}{s\hat{\alpha}_1} - \frac{c\hat{\alpha}_1}{s\hat{\alpha}_1}\hat{F}_{z2} \tag{23}$$

$$\begin{aligned}\hat{F}_{x3} = {} & \frac{s\hat{\alpha}_2}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z1} \\ & - \frac{s\hat{\alpha}_1 c\hat{\theta}_2 c\hat{\alpha}_2 + s\hat{\alpha}_2 c\hat{\alpha}_1}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} + \frac{c\hat{\theta}_2}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{24}$$

$$\hat{F}_{y3} = \frac{\hat{F}_{z2}}{s\hat{\alpha}_2} - \frac{c\hat{\alpha}_2}{s\hat{\alpha}_2}\hat{F}_{z3} \tag{25}$$

$$\begin{aligned}\hat{F}_{x4} = {} & \frac{c\hat{\theta}_3}{s\hat{\alpha}_1 s\hat{\theta}_2}\hat{F}_{z1} \\ & - \frac{c\hat{\theta}_3 c\hat{\theta}_2 s\hat{\alpha}_1 c\hat{\alpha}_2 + c\hat{\theta}_3 s\hat{\alpha}_2 c\hat{\alpha}_1 - s\hat{\theta}_3 s\hat{\alpha}_1 s\hat{\theta}_2}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} \\ & - \frac{s\hat{\theta}_3 s\hat{\theta}_2 c\hat{\alpha}_2 - c\hat{\theta}_3 c\hat{\theta}_2}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{26}$$

$$\hat{F}_{y4} = -\frac{s\hat{\alpha}_2 c\hat{\alpha}_3 s\hat{\theta}_3}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z1} \tag{27}$$

$$\begin{aligned}& + \frac{c\hat{\alpha}_3 s\hat{\theta}_3 c\hat{\theta}_2 s\hat{\alpha}_1 c\hat{\alpha}_2 + c\hat{\alpha}_3 s\hat{\theta}_3 s\hat{\alpha}_2 c\hat{\alpha}_1 + c\hat{\alpha}_3 c\hat{\theta}_3 s\hat{\alpha}_1 s\hat{\theta}_2}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} \\ & - \frac{c\hat{\alpha}_3 s\hat{\theta}_3 c\hat{\theta}_2 + c\hat{\alpha}_3 c\hat{\theta}_3 s\hat{\theta}_2 c\hat{\alpha}_2 - s\hat{\alpha}_3 s\hat{\alpha}_2 s\hat{\theta}_2}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{28}$$

$$\begin{aligned}\hat{F}_{z4} = {} & \frac{s\hat{\alpha}_3 s\hat{\theta}_3}{s\hat{\alpha}_1 s\hat{\theta}_2}\hat{F}_{z1} \\ & - \frac{c\hat{\theta}_2 s\hat{\theta}_3 s\hat{\alpha}_1 c\hat{\alpha}_2 s\hat{\alpha}_3 + s\hat{\theta}_2 c\hat{\theta}_3 s\hat{\alpha}_1 s\hat{\alpha}_3 + s\hat{\theta}_3 c\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\alpha}_3}{s\hat{\alpha}_1 s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z2} \\ & + \frac{c\hat{\theta}_2 s\hat{\theta}_3 s\hat{\alpha}_3 + s\hat{\theta}_2 c\hat{\theta}_3 c\hat{\alpha}_2 s\hat{\alpha}_3 + s\hat{\theta}_2 s\hat{\alpha}_2 c\hat{\alpha}_3}{s\hat{\alpha}_2 s\hat{\theta}_2}\hat{F}_{z3}\end{aligned} \tag{29}$$

[3]The authors found that many of the algebraic expressions for the static force analysis of the RCCC linkage presented in [4] have some misprints. For this reason, all the equations have been deduced again and reported (hopefully) correct in this paper.

The modeling of friction in the kinematic pairs of a Cardan joint

In the ideal Cardan joint, the resultant of the reaction force at the kinematic pairs is zero and there is not any displacement along the joint axes. This is not necessarily true in presence of manufacturing errors. In revolute joints $\dot{s}_i = 0$, thus the effects of friction on the reaction force component F_{iz} are herein neglected. However, these can be taken into account when detailed informations on the geometry of the revolute joint are available.

Revolute pairs

The frictional forces at the ith revolute joint arise from two sources:

- reaction forces F_{xi}, F_{yi} and F_{zi};
- reaction moments M_{xi} and M_{yi}

The resistant action about the z axis, arising from the reaction moment M_{xi} and M_{yi}, can be modeled [6, 7, 8] according to the scheme presented in Figure 3.

In particular, for our purposes, the torque M_{ix} is substituted by two parallel and opposite forces F acting normally to the revolute joint axis. Because of the presence of friction, these forces generate the frictional torque

$$\tau_f^{xi} = f\frac{d_i}{L_i}M_{xi} \, , \tag{30}$$

where d_i is the diameter of the journal bearing, L_i the distance between bearing supports[4] and f the friction coefficient.

Similarly, the torque M_{yi} generate the frictional torque

$$\tau_f^{yi} = f\frac{d_i}{L_i}M_{yi} \, . \tag{31}$$

Therefore the friction reaction torque component along the z axis is computed by letting

$$M_{zi} = -\text{sign}\left(\dot{\theta}_i\right)\left(\tau_f^{(i)} + \frac{fd_i}{2}\sqrt{F_{xi}^2+F_{yi}^2}\right) \, , \tag{32}$$

where

$$\tau_f^{(i)} = f\frac{d_i}{L_i}\sqrt{M_{xi}^2+M_{yi}^2} \, . \tag{33}$$

[4]For a single support bearing, L_i represents the length of the bearing.

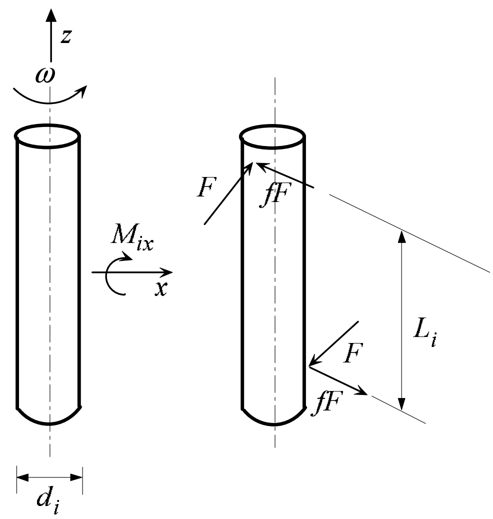

Figure 3: Modeling of friction in the revolute pairs

It is clear that the presence of friction alter the equilibrium of the links. However, because of the high efficiency values this effect is not herein considered.

Cylindric pairs

In the frictionless cylindric pair $\dot{s}_i \neq 0$, $M_{zi} = 0$ and $F_{zi} = 0$. When friction is considered the values of M_{zi} and F_{zi} must be computed. However, the computation of M_{zi} is carried out with the same model adopted for the revolute pair.

Thus, the following substitutions are required

$$F_{zi} = -\text{sign}\left(\dot{s}_i\right) f\left(\sqrt{F_{xi}^2+F_{yi}^2} + 2\frac{\sqrt{M_{xi}^2+M_{yi}^2}}{L_i}\right) \, , \tag{34}$$

$$M_{zi} = -\text{sign}\left(\dot{\theta}_i\right)\left(\tau_f^{(i)} + \frac{fd_i}{2}\sqrt{F_{xi}^2+F_{yi}^2}\right) \, . \tag{35}$$

1 Numerical example

The average efficiency η_m of a Cardan joint, according to the model described in [8] can be obtained using the chart of Figure 5, where $a = 2fd_i/(\pi L_i)$ is an adimensional parameter introduced by A. Morecki [5]. In the mentioned model, the Cardan joint has no manufacturing tolerances and the energy losses are computed including all the four revolute joints.

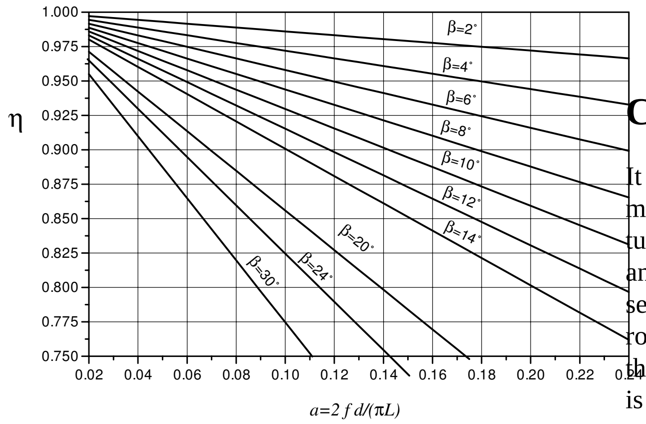

Figure 4: Average efficiency in a Cardan joint without manufacturing tolerances [8]

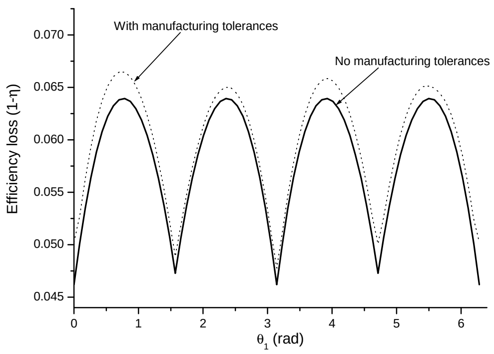

Figure 5: Instantaneous efficiency loss in a Cardan joint with and without manufacturing tolerances

The present analysis compares the instantaneous efficiency loss $(1-\eta)$ for two cases:

- No manufacturing tolerances (*i.e.* $a_j = 0$, $(j = 1,2,3,4)$.

- Presence of manufacturing tolerances, estimated in the following values of axes offsets $a_1 = a_2 = a_3 = a_4 = 0.5$. Angular errors on α_j could be included as well.

The remaining geometric parameters are as follows:

- $\alpha_1 = \alpha_2 = \alpha_3 = 90°$, $\alpha_4 = 150°$, L_i=50, d_i=40, $(i = 2,3,4)$, f=0.05.

Conclusions

It has been proposed a formulation for computing the mechanical efficiency of a Cardan joint with manufacturing tolerances or mounting errors. The model foresee an increase of energy losses due to presence of axes offsets. These losses can be explained because of the reciprocating relative motion between adjacent links along the shaft axes. In the ideal Cardan joint such motion is absent. Current investigations carried out by this research unit include experimental tests and and theoretical of inertia effects and operating parameters .

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Appendix

Fortran procedure for computing the ATAN2 function with dual numbers as arguments. This procedure requires those in the module `dualnumbers` implemented by D.E. Fasse [9].

```
function dual_atan2(A,B) result(C)
type(dual),intent(in) :: A,B
type(dual):: C
type(dual)::A1,B1
real*8 :: eps,x1,y1,x,y
eps=1.d-10
A1=A;B1=B
x=A1%pr/B1%pr
y=(A1%du*B1%pr-A1%pr*B1%du)/&
(B1%pr*B1%pr)
x1=B1%pr;y1=A1%pr
if ((abs(x1)<eps).AND.(y1>0.)) then
C%pr=pi/2.;C%du=0.
return
end if
if ((abs(x1)<(eps)).AND.(y1<0.)) then
C%pr=3*pi/2.;C%du=0.
return
end if

! Quadrant n.1
if ((x1>0.).AND.(y1>0.)) then
C%pr=atan(x);C%du=y/(1+x**2)
return
end if

! Quadrant n.2 and 3
if (x1<0.) then
C%pr=atan(x)+pi;C%du=y/(1+x**2)
return
end if
! Quadrant n.4
if ((x1>0).AND.(y1<0)) then
C%pr=atan(x)+2*pi;C%du=y/(1+x**2)
return
end if

end function dual_atan2
```

Acknowledgement

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