Research and Analysis

The Carnot and Amin Cycles

Nicolas Leonard Sadi Carnot (1796-1832) was responsible for developing the model for what is the most thermal efficient cycle possible, now known as the Carnot cycle. The cycle is the foundation of all heat devices; it was the basis behind the engine for the car, the fridge, the jet plane, the lawn mower and many more. The cycle consists of four basic reversible processes and is reversible as a whole. It results in no change in entropy and is therefore only hypothetical as all physical processes involve some change in entropy. The four stages are:

1. Heat transfer from the working fluid to the low-temperature reservoir, an isothermal compression.
2. Adiabatic increase in the temperature of the working fluid, an adiabatic compression.
3. Heat transfer from the high-temperature reservoir to the working fluid, an isothermal expansion.
4. Adiabatic decrease in the temperature of the working fluid, an adiabatic expansion.

Definitions:

Isothermal: a change that takes place at a constant temperature.

Adiabatic: a change that takes place without any energy being exchanged by heating.

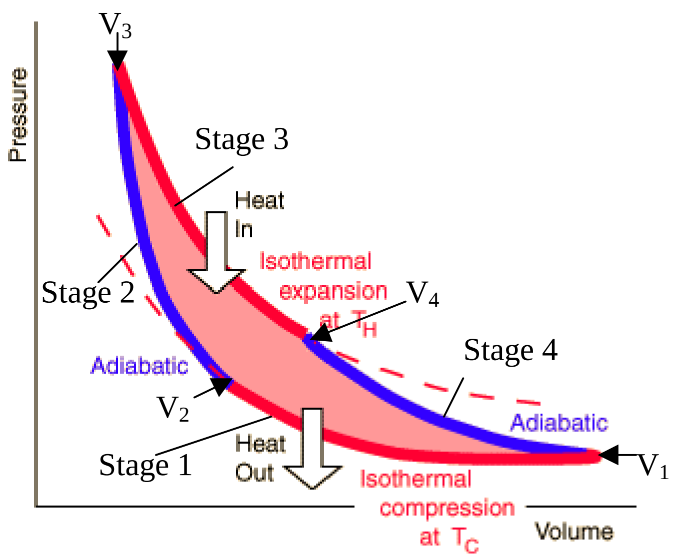


Figure 1: The Carnot cycle.[1]

Carnot was born in 1796 in the Palais du Petit-Luxembourg into a time of great political unrest in France. He was educated up to the age of 16 years by his father before being accepted into the Ecole Polytechnique in Paris. After graduating, Carnot gained qualifications in military engineering. Unfortunately for him, however, his father was exiled when Napoleon was defeated, because he held a prominent position of office under Napoleon, and Carnot himself was unable to gain any promotions or respect in the army. He left the army, joined the General Staff Corps, took leave on half pay and began to attend courses at various Parisian institutions. He took a specific interest in steam engines and began research which led to the mathematical theory of heat and the beginning of modern ideas for the theory of thermodynamics. Despite achieving many proofs and formulating many ideas, Carnot only published one work, a book called *Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* in 1824, which includes the description of the 'Carnot cycle'. It tried to answer two fundamental questions, 'is there and upper limit to the power of heat?' and 'is there a better means than steam to produce this power?'. Carnot continued working until 1827, when he was called up to full time duty by the General Staff Corps. He retired in less than a year and returned to Paris to focus on his research. He was distracted, however, by the political situation of the country and became active in supporting the republic and improving public education. In 1832, he was taken ill and then contracted cholera. He died at the age of just 36 but his ideas on steam and the theory of thermodynamics survived despite never having been published. They contributed to the development of the second law of thermodynamics by Clausius and Thomson and also prompted Joule to perform certain experiments on the friction of gases 20 years later.[2]

Carnot derived the following formula for the efficiency of a Carnot engine,

$$\eta = 1 - \frac{T_C}{T_H}$$

T_C = Initial temperature
T_H = Highest temperature reached in the cycle

I will first consider the physics behind the model and show how it achieves such thermal efficiency.

Firstly there is an isothermal compression. This means that there is no change in temperature (T and $\Delta T = 0$) and therefore the internal energy (U) of the gas must also remain constant ($\Delta U = 0$). The gas is having work done to it as a result of compression, and as the temperature remains constant, it must be cooled, releasing energy to the surroundings. If Q = energy supplied by heating and W = the work done by the gas, then $\Delta Q = \Delta U + \Delta W$ and the internal energy of the gas is constant, $\Delta Q = \Delta W < 0$.

Next the gas goes through an adiabatic compression. This means that there is no heating of the gas during the process. Therefore ΔQ must = 0. Thus the work done to the gas results in it changing temperature. If there is an adiabatic compression, the gas must heat up ($\Delta W < 0$), and as $\Delta Q = \Delta U + \Delta W$ and $\Delta Q = 0$, $\Delta U > 0$, the gas heats up.

In the third stage, there is an isothermal expansion. Here the gas is doing the work, and as the temperature remains constant, it takes the energy from the surroundings, some of which had originally been created by isothermal compression in the first stage.

Finally there is an adiabatic expansion. The energy supplied is 0 and therefore the gas must cool down as it is doing work. This brings us back to the first stage in the cycle. (References: [1, 4 & 10])

Now let us add some figures to the model to obtain a value for the efficiency to show that the formula works. For argument's sake, initial temperature (T_C) = 300K and the highest temperature (T_H) = 385K and we are considering 1 mole of a monatomic gas.

$$pV = nRT$$

p = pressure
V = volume
n = number of moles
R = molar gas constant
T = temperature

In the first stage (isothermal compression) the temperature is constant, therefore pV is also constant. The gas is having work done to it and therefore the energy of the surroundings is heating up. The work done (ΔW) = area underneath the curve of stage 1 (figure 1). Let initial volume (V_1) be 10.0 x 10^{-4} m^3 (p = 2500 kPa) and final volume at the end of stage 1 (V_2) be 6.00 x 10^{-4} m^3 (p = 4160 kPa).
Area under the curve = $\int nRT/V \, dV$ (from pV = nRT and the graph plots p against V)**[9]**
Work done = [nRT lnV] between V_1 and V_2.
Work done =

Therefore work done =

In the second stage (adiabatic compression), the gas heats up to 385K, the volume decreases to 4.00 x 10^{-4} m^3, and as it is compressed the pressure will rise to 8010 kPa.
$PV^\gamma = k$ (constant). For proof see Appendix 1.
$\gamma = C_p/C_v = 2.5R/1.5R = 5/3$ for a monatomic ideal gas. For proof see Appendix 2.
C_p = molar heat capacity of a gas at a constant pressure.
C_v = molar heat capacity of a gas at a constant volume.
Work done = area under curve for stage 2 (figure 1) = $\int k/V^\gamma \, dV$.
Work done = $k[\{1/(-\gamma-1)\} \times \{1/(V^{\gamma-1})\}]$ between V_2 (initial volume for stage 2) and V_3 (final volume for stage 2)
Therefore: Work done =

Work done =

Work done =

Energy absorbed ($\Delta Q = \Delta U$) = $nC_v\Delta T$ (Proof of $\Delta Q = \Delta U$ for a Carnot Cycle is beyond A-level and I could not find a proof from any source).
ΔQ =

ΔQ =

In the third stage (isothermal expansion) work is being done by the gas for the first time. Temperature must remain at 385K as the volume increases to 8.00×10^{-4} m^3 (p = 4000 kPa). So as in the first stage:
Area under the curve of stage 3 (figure 1) = $\int nRT/V \, dV$ (from pV = nRT and the graph plots p against V)
Work done = [nRT lnV] between V_4 (final volume for stage 3) and V_3 (initial volume for stage 3)
Work done =

Work done =

Finally there is adiabatic expansion to the original temperature of 300K at 2500 kPa and 10.0×10^{-4} m^3. Work done is calculated as in stage 2:
$PV^\gamma = k$ (constant). For proof see Appendix 1.
Work done = area under curve of stage 4 (figure 1) = $\int k/V^\gamma \, dV$
Work done = $k[\{1/(-\gamma-1)\}x\{1/(V^{\gamma-1})\}]$ between V_1 (final volume for stage 4) and V_4 (initial volume for stage 4).
$\gamma = C_p/C_v = 2.5R/1.5R = 5/3$. For proof see Appendix 2
Therefore: Work done =

Work done =

Work done =

Energy absorbed ($\Delta Q = \Delta U$) = $nC_p\Delta T$ (Proof of $\Delta Q = \Delta U$ beyond my capabilities)
ΔQ =

ΔQ =

N.B. All data correct to 3 significant figures.

The thermal efficiency (η) = work done by the engine / energy absorbed by heating
η =

If this was calculated using Carnot's equation:

$$1 - (T_C/T_H) \times 100\% = 1 - (300K/385K) \times 100\% = 22\%$$

Carnot's equation shows that for a 100% efficient engine, T_C needs to be as low as possible and T_H needs to be as high as possible. The Carnot Cycle is based on there being no change in entropy, this allows it to be so efficient but also makes it theoretical. Entropy is a measure of the disorder of a system. For a reversible process change in entropy is 0 and for an irreversible process the change in entropy is positive. The change in entropy can never be negative and this forms the basis of the second law of thermodynamics. As the Carnot Cycle models a heat engine, it can never be fully reversible and therefore, in practical situations, change in entropy can never be 0, but it is for the theoretical cycle.[4]

Firstly, the proof that there is no change of entropy in the Carnot cycle.[3]
$\Delta S = \Delta Q/T$ (formula for the change of entropy in a reversible system) where S = entropy.
Integrating both sides: S = [Q/T] between two limits.
Therefore in both of the adiabatic processes, there is no heat absorbed. Therefore $\Delta Q = 0$ and ΔS also equals 0. In both of the isothermal stages, the temperature remains constant.
$Q_1/Q_2 = T_1/T_2$ (For proof see below *Proof of the efficiency of a Carnot engine*).
Q_1 = internal energy in stage 1
Q_2 = internal energy in stage 3
T_1 = temperature in stage 1
T_2 = temperature in stage 3
Therefore $Q_1/T_1 - Q_2/T_2 = 0$.
$\Delta S_1 = \Delta Q_1/T_1$
$\Delta S_2 = \Delta Q_2/T_2$
ΔS_1 = change in entropy in stage 1
ΔS_2 = change in entropy in stage 3
Therefore $\Delta S_1 + \Delta S_2 = 0$
This shows that there is no change in entropy in the Carnot Cycle.

Proof of the efficiency of a Carnot engine.[4]

From the 1st law of Thermodynamics:
$W = \Delta U - Q$
ΔU = the change in the internal energy, therefore it can be written as Q_{in}.
Q = the internal energy released, therefore it can be written as Q_{out}.
$W = Q_{in} - Q_{out}$
Efficiency of a heat energy (η) = Work done/energy absorbed by heating
$\eta = (Q_{in} - Q_{out})/Q_{in}$
$\eta = 1 - Q_{out}/Q_{in}$

In stage 1:
Work done (W) = $nRT_C(\ln V_1 - \ln V_2)$
$W = nRT_C\ln(V_1/V_2)$
In stage 2:
Work done (W) = $nRT_H(\ln V_3 - \ln V_4)$
$W = nRT_H\ln(V_3/V_4)$

Where V_1, V_2, V_3, and V_4 are volumes at different stages of the Carnot Cycle (see figure 1)

As TV^{-1} = c (a constant) (For proof see Appendix 1)
Therefore: $T_HV_3^{-1} = T_CV_1^{-1}$
And $T_HV_4^{-1} = T_CV_2^{-1}$

$(V_3/V_4)^{-1} = (V_1/V_2)^{-1}$

and $\ln(V_3/V_4)^{-1} = \ln(V_1/V_2)^{-1}$
Therefore the ratio of $Q_{out}/Q_{in} = [T_C(\ln V_1 - \ln V_2)]/ [T_H(\ln V_3 - \ln V_4)] = T_C/T_H$
Therefore $\eta = 1 - Q_{out}/Q_{in} = 1 - T_C/T_H = \eta_{carnot}$

Q.E.D.

Modern Applications of the Carnot Cycle leading to the Amin Engine and further engine models.

All modern engine processes are based on the Carnot model to maximise thermal efficiency. The major problem, however, is that for an efficiency of just 50%, with a T_c of 300K, the T_h still needs to be 600K and these temperatures are very difficult and expensive to produce commercially. The cycle is still as thermally efficient as possible, however, and therefore the majority of modern engine cycles are based on the Carnot Cycle.

One type of heat cycle that has been spawned from the Carnot Cycle is the recently developed Amin Cycle.[8] Sanjay Amin spotted that the Carnot Cycle fails to take gravity into account. He soon found that this led to an extension of the Carnot Cycle formula to a new universal formula that accounts for the gravity of the area. The Cycle is again theoretical but provides a basis for more efficient engines to be built which compensate for the changes in efficiency due to gravity. It looks as if the cycle will revolutionise the world of the motor as Carnot managed 150 years earlier:

$$\eta_{amin} = 1 - (T_C/T_H) \times e^{-mh/k(g/Tc - g/Th)}$$

T_C = initial temperature
T_H = highest temperature reached in the cycle
m = mass of the gas
h = height of the piston
k = Bolztmann constant
g = gravitational constant

Therefore, if g = 0, $\eta_{amin} = 1 - T_C/T_H = \eta_{carnot}$

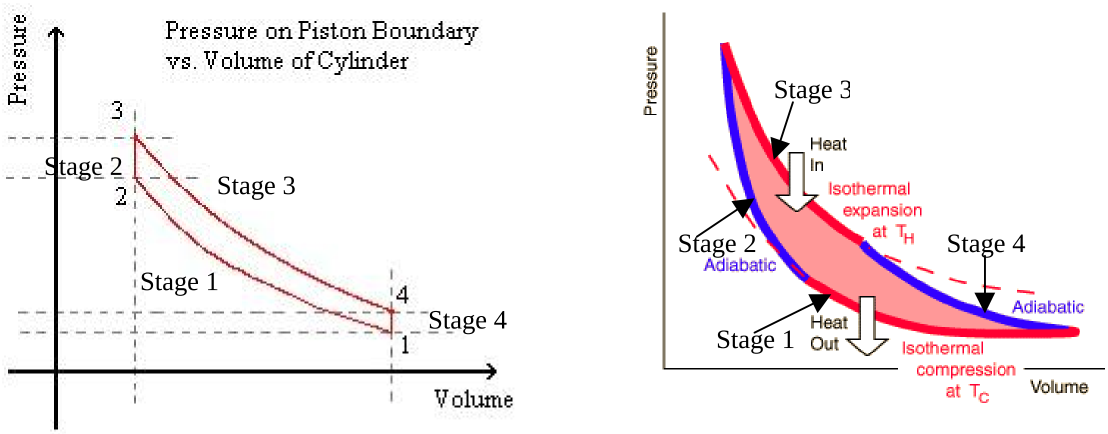


Figure 2: The Amin Cycle[5] (left) with the Carnot Cycle[1] (right) for comparison.

Unlike the Carnot Cycle, the Amin Cycle does not have any adiabatic stages. In this case pressure is increased when the angular velocity of the piston and the cylinder within the engine increases until a sufficiently higher temperature is reached. The other two stages of the cycle are isothermal as in the Carnot Cycle.

The proof of this follows from the proof of the efficiency of the Carnot Cycle but goes into mathematics well beyond that required for A-level. The four stages of the cycle are as follows:

Stage 1 – Isothermal expansion at T_C in the cylinder at low (angular velocity), $_1$.
Stage 2 – $_1$ increased to $_2$ (high) at constant volume.
Stage 3 – Isothermal compression at T_H at high $_2$.
Stage 4 – $_2$ decreased to $_1$ at constant volume.

= angular velocity

The new Amin Cycle is controversial and, according to some scientists, is not valid. The main issue of concern is that it runs on energy from the surroundings and therefore breaks the second law of thermodynamics. Dieter Bauer[6] has shown that when the function is plotted on an S (entropy) – T (temperature) graph, the line merely goes backwards and forwards as the cycle goes round. Therefore there is no area between the lines and no resultant work done. This would suggest that the effect of any work being done by the engine is cancelled by work done to it, but this can only be the case if the cycle is reversible. Although the Carnot Cycle is reversible, the Amin Cycle is not and therefore it appears that no work is done. The cycle, according to Bauer, is flawed. Other scientists have expressed similar views, for example, David Howe[7] suggested that although the energy used to speed up the system and cylinder is recovered when it slows down, the same does not happen to the gas inside the system. The site from which much of this information was taken (www.phact.org) directly criticises the site that holds the rights to the Amin Cycle (www.entropysystems.com). It accuses Entropy Systems of either fraud or ignorance as the company continues to attract investment ($1.6 million to date). Thus what is potentially the most exciting recent development in the history of the development of the heat engine is flawed. I have been in e-mail correspondence with Mr Bob Paddock (Appendix 3), who originally had the Amin Cycle posted on his web news group page (www.csonline.net/bpaddock), and after the initial enthusiasm even he has his doubts about the cycle. He has had no interaction from Mr Sanjay Amin for an extended period and I have also been unable to contact him directly.

Although he is a brilliant physicist, Sanjay Amin's cycle, originally published in 1994, appears to be invalid, although he has not so far conceded this.

The heat engine has been intricately woven into society for over 100 years and with the current concerns about the environment, efficiency is becoming an increasingly greater priority. Sadi Carnot's influence on the design and principles behind a heat engine had been bettered by none, until the Amin Cycle was discovered. Sanjay Amin's cycle is both controversial and potentially revolutionary. Even if it is eventually proved wrong, I think that benefits can be drawn from it and engine efficiency can be regarded in a new light. His insight, which may still hold a precious key to increased efficiency, has shaken up research in the area and will hopefully accelerate development in more efficient and therefore less energy consuming heat engines.

Appendices

Appendix 1

Proof that TV^{-1} = a constant and that pV = a constant (ref: [9])

First Law of Thermodynamics:

$\Delta Q = \Delta U + \Delta W$

$-\Delta U = \Delta W$

For a constant volume process, there is no work done and therefore $\Delta Q = \Delta U$.

$C_v = \Delta Q/(n\Delta T)$

$C_v = \Delta U/n(\Delta T)$

$C_v n\Delta T = \Delta U$

$0 = C_v n\Delta T - \Delta U$

$0 = C_v n\Delta T + p\Delta V$ ($\Delta W = p\Delta V$)

$0 = C_v n\Delta T + nRT\Delta V/V$ ($pV = nRT$)

$0 = C_v\Delta T/T + R\Delta V/V$

$0 = C_v\Delta T/T + (C_p - C_v)\Delta V/V$ ($C_p = C_v + R$)

$0 = \Delta T/T + (- 1)\Delta V/V$ ($= C_p/C_v$)

$-\Delta T/T = (- 1)\Delta V/V$

Integrate both sides:

$-\ln T = (- 1)\ln V$

$-\ln T = \ln V^{(- 1)}$

$\ln TV^{(- 1)} = 0$

$TV^{(- 1)} = 1$

This is constant, $TV^{(- 1)} = k$ (Where k is a constant)

Q.E.D.

$pV = nRT$

$pV/nR = T$

$(pV^*V^{(- 1)})/(nR) = k$ ($T = k/ V^{(- 1)}$)

$pV = knR$

As n, R and k are all constants, then:

$PV = c$

Q.E.D.

Appendix 2

Proof that for an ideal monatomic gas, $\gamma = 5/3$ (ref: [10])

$pV = nRT$ - equation 2.0

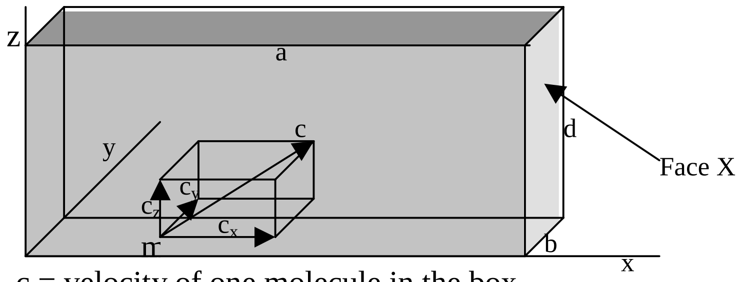
$pV = (m/M)^*RT$ m = mass of gas, M = the molar mass

$pV = (N/N_A)^*RT$ N = number if molecules, N_A = the Avogadro constant

$pV = NkT$ $k = R/N_A$ where k = Boltzmann constant - equation 2.1

also:

consider molecules in a box



c = velocity of one molecule in the box.

First, consider the movement of the molecule in the x direction:

The time taken for the molecule to reappear each time at face X (collisions are all elastic) = distance / speed = $2a/c_x$

Change in momentum = mv – mu $\quad\quad$ $v = -c_x$, $u = c_x$

P = $-2mc_x$

P/ t = $-2mc_x/(2a/c_x) = -mc_x^2/a$ = force on the molecule = -force on the wall.

N molecules in the box, therefore force on wall (F) = Nmc_x^2/a

(c_x^2 = the average of the squares of all the speeds of all the molcules)

$c_x^2 = c_y^2 = c_y^2 = (1/3)c^2$

Therefore F = $(Nmc^2)/(3a)$

p = F/area = F/bd = $Nmc^2/3abd$

= $Nmc^2/3V$ - equation 2.2

From equations 2.1 and 2.2:

$Nmc^2/3 = NkT = pV$

$mc^2/2 = 3kT/2$

Therefore the average K.E. of each molecule of an ideal gas:

= 3kT/2 - equation 2.3

U = N(3kT/2) = 3nRT/2 - equation 2.4 $\quad\quad$ (from equation 2.0 and 2.3)

$C_v = \Delta Q/(n\Delta T)$ (constant volume)

$C_p = \Delta Q/(n\Delta T)$ (constant pressure)

For a constant volume process, there is no work done and therefore $\Delta Q = \Delta U$.

$\Delta U = 3nR\Delta T/2 = \Delta Q$ $\quad\quad$ (from equation 2.4)

Therefore $C_v = \Delta Q/(n\Delta T) = 3nR\Delta T/2(n\Delta T)$

$C_v = 3R/2$ – equation 2.5

Also, $C_p = C_v + R$,

Therefore, $C_p = (3R/2) + R$ $\quad\quad$ (from equation 2.5)

$C_p = 5R/2$

Therefore, for an ideal monatomic gas, $\gamma = C_p/C_v = 5/3$

Q.E.D.

Appendix 3
E-mail correspondence with Bob Paddock.

On 07/11/00, Adam Briggs wrote:
I am currently at school working on a Physics Research and Analysis Project for my Physics A-level. Initially I decided to work on the Carnot Cycle but soon found that I wanted to develop the work to something much more modern and this was how I came across the Amin Cycle.

I have taken all the information I can from the Entropy Systems Inc. web site but would be extremely grateful if you could write to me and explain, in simple Layman's terms, how the cycle achieves such high efficiency, it is of considerable interest me.

On 09/11/00, Bob Paddock replied:
Did you also look at http://www.oilcity.org/research/ ?

I'm not sure I understand it well enough my self to do that. Amin said he was going to post his book that does explain it well. I'll see if I can find the URL.

The simple explanation is that the standard Carrnot Cycle does not take Gravity into account, where Amin does.

On 09/11/00, Adam Briggs wrote:
Thank you for your prompt reply. I have looked at oilcity.org and the information there is certainly useful. I just have a few more questions however. When did Mr Amin develop this cycle?, can engines now be designed to compensate for this gravitational effect therefore making their efficiency greater?, is his cycle just theoretical in the same way that Carnot's was? and finally what is your relationship with Mr Amin (assuming you allow me to quote your name in my references)?

On 11/11/00, Bob Paddock replied:
His book was published in Sept. of 1994. I know he was working on it as early as 1987, probably before that.

The gravity effects being accounted for over Carnot is what makes the "Amin Cycle" more efficient than Carnot, at least according to Amin.

I saw his first device, the one based on air conditioning. It did exactly what he said. It would reduce the output temperature by 30 degrees from the input temp. I don't recall if that was 'F or 'C, if I had to guess I'd say it was 'C. I've never seen his engine, which I do find a bit odd, see next question.

Long before Internet as every one knows it today, not so long ago (~1985) I ran a "Bulletin Board System" (BBS).

I had a lot of the alternative energy files from the Keelynet System (http://www.keelynet.com see down load section) on my BBS.

Amin went school near by, so he enjoyed hanging out on the BBS checking out the obscure files. I'm sure some of those files aided him in his ideas.

Entropy Systems is about 50 miles from me now, but he never seems to find time to have me come over and check out his latest stuff. I find this odd, but I'll leave it at that.

You can quote me if you wish.

Also make sure you check out the 'other side'...
In the interest of balance Eric Krieg requested I add a link to his site, http://www.phact.org/e/z/amin.htm.

On 11/11/00, Adam Briggs wrote:
Thanks, this is certainly very interesting and useful stuff.
What is your opinion on it all??

On 13/11/00, Bob Paddock replied:

I have not doubt that Amain is a technically competent person. Personally I would have done some of the business affairs differently, but thats easy to say in hind-sight.
One thing you can say is that he had the guts to take on the Guardians of Status Quo (see the Cold Stone quote at the end of my home page).

Amin has over looked the old work of Viktor Schauberger, that just might have let him over come "the problem of friction". Do a web search and you should come up with some interesting info.

What I've not seen in the ones I've read was the fact that VS's devices only worked at certain critical temperatures, which you find in the original German language papers (or translations of).
This indicates some type of phase change is taking place in their operation.

Learn what every one has to teach you, but always keep an open mind to other possibilities...

References

1. C. R. Nave. 2000. HyperPhysics - *The Carnot Cycle*. http://hyperphysics.phy-astr.gsu.edu/hbase/thermo/carnot.html
   (This was a well written site which gave useful information on almost any physics related issue. There was a great diagram of the Carnot Cycle which I decided to use. Other web sites such as ref: [2] gave similar diagrams but they were not as colourful and clear.)
2. University of Queensland, Department of Mechanical Engineering. *The Carnot Cycle*. http://www.uq.edu.au/~e4nsrdja/teaching/e4213/Related/Cycles/Carnot.htm
   (Simple explanations of various physical concepts, including the Carnot Cycle. Part of a university web site and very similar to many sites visited.)
3. G. R. Noakes. 1960. *New Intermediate Physics*. Macmillan.
   (I found a variety of sources that explained the zero change in entropy in the Amin Cycle but they all seemed to use slightly different methods for the proof, leading to confusion, e.g. R. Muncaster. 1989. *A-level Physics (3rd edition)*. ST(P). Noakes explains the theory behind it the clearest.)
4. P. A. Tipler. 1999. *Physics, 4th edition*. W. H. Freeman.
   (This was a good quality modern textbook that gave a very clear outline of the proof of the Carnot Cycle, one of very few books that actually presented a proof.)
5. S. Amin. 1996. *Entropy – The key to Unlimited Resources: Amin's Cycle*. http://www.oilcity.org/research/amin/amincyc.htm
   (Sanjay Amin's original article which obviously outlines his arguments. If is more of a theoretical approach, however, than other explanations so there were less proofs but more explaining.)
6. W. D. Bauer. 1999. *Amin Cycle errors?* http://www.escribe.com/science/freenrg/m5098.htm
   (A single article analysing Amin's Cycle. It is written in mainly laymen's terms and therefore is easier to understand than, say, some of the other articles found at phact.com.)
7. D. Howe. *Free energy claims of Entropy Systems Inc. – Sanjay Amin.* http://www.phact.org/e/z/amin.htm
   (This site contained numerous articles criticising the Amin Cycle but explanations for it vary in clarity. It is the only site that I found of this kind.)
8. Entropy Systems, Inc. News & Events. http://www.entropysystems.com
   (The web site for the company that bought the Amin Cycle and for whom Amin works. The company has the sole rights to the cycle and also gives explanations of how the cycle works. It is not as clear, however, as Ref: [5])
9. D. Tabor. 1969. *Gases, liquids & Solids*. Penguin.
   (An old but well written analysis of thermodynamics which gives comprehensive proofs for many problems. I could not, for example, find this particular proof in Ref: [4])
10. Akrill, Bennet & Millar. 1994. *Physics – second edition*. Hodder & Stoughton.
   (A further physics textbook aimed at the A-level student. It therefore does not take proofs and laws as far as many of the older books (i.e. Ref: [3] or Ref: [9]) but it is very well written.)

I found that the sources I used were relatively consistent with each other. Some of the older books gave proofs of certain formulae but used different terminology and symbols to put across their point. This made further research necessary but I was able, in all cases, to find an argument which I could follow through and understand myself and therefore portray in a clear manner to those reading this document. An example of this is the proof in Appendix 2.